Decoupling limits of $\mathcal{N} = 4$ super Yang-Mills on $\mathbb{R} \times S^3$

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ABSTRACT: We find new decoupling limits of $\mathcal{N} = 4$ super Yang-Mills (SYM) on $\mathbb{R} \times S^3$ with gauge group $\mathrm{SU}(N)$. These decoupling limits lead to decoupled theories that are much simpler than the full $\mathcal{N} = 4$ SYM but still contain many of its interesting features. The decoupling limits correspond to being in a near-critical region, near a point with zero temperature and critical chemical potentials. The new decoupling limits are found by generalizing the limits of hep-th/0605234 to include not only the chemical potentials for the SU(4) R-symmetry of $\mathcal{N} = 4$ SYM but also the chemical potentials corresponding to the SO(4) symmetry. In the decoupled theories it is possible to take a strong coupling limit in a controllable manner since the full effective Hamiltonian is known. For planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ all the decoupled theories correspond to fully integrable spin chains. We study the thermodynamics of the decoupled theories and find the Hagedorn temperature for small and large values of the effective coupling. We find an alternative formulation of the decoupling limits in the microcanonical ensemble. This leads to a characterization of certain regimes of weakly coupled $\mathcal{N} = 4$ SYM in which there are string-like states. Finally, we find a similar decoupling limit for pure Yang-Mills theory, which for the planar limit leads to a fully integrable decoupled theory.

KEYWORDS: Strong Coupling Expansion, AdS-CFT Correspondence, Bethe Ansatz, Supersymmetric gauge theory.

Contents

1. Introduction

The AdS/CFT correspondence conjectures a precise duality between $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory and type IIB string theory on $AdS_5 \times S^5$ [1–3]. As a consequence of this correspondence, it is believed that weakly coupled string theory on $AdS_5 \times S^5$ emerges from large N $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ in the limit of

large 't Hooft coupling. This is in accordance with the ideas of 't Hooft of the emergence of string theory from gauge theory when the number of colors is sent to infinity [4].

However, taking the strong 't Hooft coupling limit of large N SU(N) $\mathcal{N} = 4$ SYM is a highly non-trivial task. For planar $\mathcal{N} = 4$ SYM, significant progress has been made, in particular with the idea of integrable spin chains as being the connecting link between gauge theory and string theory [5]. However, despite the remarkable progress, it seems a highly difficult task to use this to understand $\mathcal{N} = 4$ SYM beyond the planar diagrams, and it is furthermore difficult to generalize the methods to other gauge theories.

In this paper, we take a different route, following the papers [6–9]. The idea is to consider decoupling limits of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with gauge group SU(N). By taking such a decoupling limit, the remaining decoupled theory is significantly simpler than the full $\mathcal{N} = 4$ SYM theory and this makes it possible to take a strong coupling limit of the decoupled theory in a controllable manner.

The decoupling limits are taken by considering the partition function in the grand canonical ensemble, which depends on the temperature and the chemical potentials. The chemical potentials are ω_1 and ω_2, corresponding to the two charges S_1 and S_2 of the SO(4) group of S^3, and Ω_1, Ω_2 and Ω_3 corresponding to the three R-charges J_1, J_2 and J_3. The idea is to consider the behavior of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ near a critical point of zero temperature and critical chemical potential $(\omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3) = (n_1, n_2, n_3, n_4, n_5)$, with n_i being fixed numbers. Writing then $(\omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3) = (n_1\Omega, n_2\Omega, n_3\Omega, n_4\Omega, n_5\Omega)$, with Ω a parameter ranging from 0 to 1, the decoupling limits of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ take the form

$$\Omega \to 1, \quad \tilde{T} \equiv \frac{T}{1 - \Omega} \text{ fixed}, \quad \tilde{\lambda} \equiv \frac{\lambda}{1 - \Omega} \text{ fixed}, \quad N \text{ fixed}, \tag{1.1}$$

where λ is the 't Hooft coupling. In such a decoupling limit we show that the full partition function of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ reduces to ($\tilde{\beta} = 1/\tilde{T}$)

$$Z(\tilde{\beta}) = \text{Tr}\left(e^{-\tilde{\beta}(D_0 + \tilde{\lambda}D_2)}\right), \tag{1.2}$$

where the trace runs over a subset of the states, and the D_0 and D_2 operators come from the weak coupling expansion of the dilatation operator $D = D_0 + \lambda D_2 + \mathcal{O}(\lambda^{3/2})$, with D_0 being the bare scaling dimension and D_2 the one-loop contribution. The trace in eq. (1.2) runs over the subset of states corresponding to the set of gauge-invariant operators of $\mathcal{N} = 4$ SYM fulfilling the equation $D_0 = J$, with $J \equiv n_1 S_1 + n_2 S_2 + n_3 J_1 + n_4 J_2 + n_5 J_3$.

The general idea with these decoupling limits is then that the full $\mathcal{N} = 4$ SYM reduces to a subsector, and that the full effective Hamiltonian reduces to the truncated Hamiltonian $D_0 + \tilde{\lambda} D_2$ containing only the zero and one-loop terms. This makes it possible to take the large $\tilde{\lambda}$ limit. Since for an expansion for small $\tilde{\lambda}$ a contribution at order $\tilde{\lambda}^n$ origins from a λ^n term in the full theory we can in this sense say that $\tilde{\lambda} \to \infty$ corresponds to taking a strong coupling limit of the theory, even though λ is small in the limit (1.1). Therefore, we are able to take explicitly a strong coupling limit by selecting only a subclass of the diagrams for the full theory.

A particular limit of the above kind was found and studied in [6, 7] with the critical point given by $(n_1, n_2, n_3, n_4, n_5) = (0, 0, 1, 1, 0)$. In the limit (1.1) all the states decouple except for those in the SU(2) sector. For the single-trace operators of planar $\mathcal{N} = 4$ SYM, the $\tilde{\lambda}D_2$ term corresponds to the Hamiltonian of the ferromagnetic $XXX_{1/2}$ Heisenberg spin chain. Therefore, weakly coupled planar $\mathcal{N} = 4$ SYM becomes equivalent to the Heisenberg spin chain in this decoupling limit. In [7] this was used to find the spectrum in the limit of large $\tilde{\lambda}$. The spectrum for $\tilde{\lambda} \to \infty$ was shown to be given by the spectrum of free magnons in the Heisenberg spin chain.

The AdS/CFT correspondence states that planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ is dual to tree-level type IIB string theory on $\mathrm{AdS}_5 \times S^5$. Thus, the decoupling limit (1.1) with $(n_1, n_2, n_3, n_4, n_5) = (0, 0, 1, 1, 0)$ of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ is dual to the corresponding decoupling limit of tree-level string theory on $\mathrm{AdS}_5 \times S^5$ [7]. By employing a certain Penrose limit [10], we found in [7] the spectrum for large $\tilde{\lambda}$ and matched this to the spectrum found on the gauge theory side, for large $J = J_1 + J_2$. We furthermore used this to match the Hagedorn temperature as computed on the gauge theory and string theory sides. The match of the spectrum and the Hagedorn temperature means that the strong coupling limit $\tilde{\lambda} \to \infty$ on the gauge theory side correctly matches the same decoupled regime in string theory. Therefore, the decoupling limit (1.1) provides us with a precise way to match gauge theory with string theory.

In this paper we find all the decoupling limits of the form (1.1), where $(\omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3) = (n_1, n_2, n_3, n_4, n_5)$ corresponds to a critical value for the chemical potentials of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. We find a total of fourteen such decoupling limits of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$, three of them found previously in [6]. These fourteen limits correspond to fourteen different subgroups of the total symmetry group $PSU(2, 2|4)$ of $\mathcal{N} = 4$ SYM. We show that in the planar limit, each of the fourteen decoupled theories corresponds to a fully integrable spin chain (previously considered in [11]). Some of these decoupled theories are well-known theories in the Condensed Matter literature, thus in this sense we have found limits of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ where it reduces to known Condensed Matter theories. However, when going beyond the planar part of $\mathcal{N} = 4$ SYM, the decoupling limits give rise to new decoupled theories.

Of the fourteen decoupling limits that we find, two give rise to trivial decoupled theories. The remaining twelve non-trivial decoupled theories are divided into nine theories with scalars and three without scalars. We explain that the presence of scalars is crucial for how the theory behaves in the large $\tilde{\lambda}$ limit. One of the theories with scalars has a SU(1, 2|3) symmetry, and we show that all the other decoupled theories can be seen to be a subsector of the theory with SU(1, 2|3) symmetry.

We consider in detail the decoupled theories in the planar limit. We employ recent results in the literature to write down the Bethe equations for the decoupled theories, and use this to find the low energy limit of the spectrum for each theory.

We analyze furthermore the thermodynamics of the decoupled theories in the planar limit. For each theory we compute the partition function and the Hagedorn temperature for zero coupling, and for small $\tilde{\lambda}$ we find the first correction in $\tilde{\lambda}$. For the nine theories with scalars we use the results for the low energy spectra to determine the Hagedorn

temperature for large $\tilde{\lambda}$. We furthermore explain why the large $\tilde{\lambda}$ behavior for the three theories without scalars is difficult to attain.

We provide an equivalent formulation of the decoupling limits (1.1) in the microcanonical ensemble, i.e. with the limits formulated in terms of D, S_1, S_2, J_1, J_2 and J_3. This is crucial for translating the limits to the string side of the AdS/CFT correspondence, but it is also highly important in order to understand precisely which regimes of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ the decoupled theories correspond to. It is furthermore a check that the decoupling limits are consistent. We find in particular that for the nine non-trivial theories with scalars, the states in $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ that dominate in the strong coupling limit $\tilde{\lambda} \to \infty$ are the ones in the regime

$$|D - J| \ll \lambda \ll 1, \quad J \gg 1 \tag{1.3}$$

with $J \equiv n_1 S_1 + n_2 S_2 + n_3 J_1 + n_4 J_2 + n_5 J_3$.

As we discuss in the paper, formulating the limits in the microcanonical ensemble also means that we can think of the limits as being taken of the gauge-invariant operators of $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$, rather than of the states of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$.

Finally, we use our insights obtained for $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ to formulate a new decoupling limit of pure Yang-Mills (YM) theory on $\mathbb{R} \times S^3$. Our new decoupling limit of pure YM shares many features with one of the decoupling limits for $\mathcal{N} = 4$ SYM, corresponding to $(n_1, n_2, n_3, n_4, n_5) = (1, 1, 0, 0, 0)$. For planar pure YM we show that the decoupled theory obtained from the decoupling limit corresponds to an integrable spin chain. We furthermore analyze the large $\tilde{\lambda}$ limit and discuss the implications for finding a string-dual of pure YM.

2. New decoupling limits

In this section we generalize the recently found decoupling limits [6] for weakly coupled $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with gauge group SU(N) to include chemical potentials for the R-charges of the SU(4) R-symmetry as well as the Cartan generators of the SO(4) symmetry group of S^3. The limits are taken of the thermal partition function of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the grand canonical ensemble and they are valid for finite N. For each decoupling limit, only a subset of the states of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ survive and the effective Hamiltonian truncates to include only the tree-level and one-loop terms of the full theory. In section 2.3 we list all of the fourteen different decoupling limits that one can have, along with the field content and the symmetry algebra for each of the decoupled theories. Finally, we show in section 2.4 that all the decoupled sectors are closed under the action of the one-loop dilatation operator D_2.

2.1 General considerations

We consider $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with gauge group SU(N). We define the 't Hooft coupling as[1]

$$\lambda = \frac{g_{\mathrm{YM}}^2 N}{4\pi^2}, \tag{2.1}$$

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[1]The $4\pi^2$ factor is included in the 't Hooft coupling for our convenience.

where $g_{\rm YM}$ is the Yang-Mills coupling of $\mathcal{N} = 4$ SYM. For $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ the states are mapped to the operators of $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$, with the energy of a state mapped to the scaling dimension of the operator (we assume here that the radius of S^3 equals one). Since we are on an S^3 we only have gauge singlet states. This means that the set of operators M that we should consider is the set of gauge invariant operators, which are all the possible linear combinations of the multi-trace operators

$$\mathrm{Tr}\left(A_1^{(1)} A_2^{(1)} \cdots A_{L_1}^{(1)}\right) \mathrm{Tr}\left(A_1^{(2)} A_2^{(2)} \cdots A_{L_2}^{(2)}\right) \cdots \mathrm{Tr}\left(A_1^{(k)} A_2^{(k)} \cdots A_{L_k}^{(k)}\right). \qquad (2.2)$$

Here $A_j^{(i)} \in \mathcal{A}$, with $\mathcal{A}$ being the set of letters which is the singleton representation of $psu(2,2|4)$. We review the set of letters $\mathcal{A}$ in detail in section 2.2. Each state carries quantum numbers according to the Cartan generators of $psu(2,2|4)$. These are the energy E, the two angular momenta S_1, S_2 corresponding to the SO(4) symmetry of S^3, and the three R-symmetry charges J_1, J_2, J_3 corresponding to the Cartan generators of the SU(4) R-symmetry subgroup of $P\mathrm{SU}(2,2|4)$. For the corresponding operator we have the scaling dimension D, along with angular momenta S_1, S_2 and the R-symmetry charges J_1, J_2, J_3.

In general, we can write the partition function of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with gauge group SU(N) in the grand canonical ensemble as

$$Z_{\lambda,N}(\beta, \omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3) = \mathrm{Tr}_M\left[\exp\left(-\beta D + \beta \sum_{a=1}^{2} \omega_a S_a + \beta \sum_{i=1}^{3} \Omega_i J_i\right)\right] \qquad (2.3)$$

where $T = 1/\beta$ is the temperature, ω_1, ω_2 are the chemical potentials corresponding to S_1, S_2, and $\Omega_1, \Omega_2, \Omega_3$ are the chemical potentials corresponding to J_1, J_2, J_3. M is the set of gauge invariant operators defined above. Note that the dependence on λ enters only through the dilatation operator D, while the N dependence enters through D and the set of operators M.

In the following we are interested in the situation in which some or all of the chemical potentials are set to be proportional to the same parameter Ω and the rest are zero. We write this in general as

$$(\omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3) = (n_1\Omega, n_2\Omega, n_3\Omega, n_4\Omega, n_5\Omega) \qquad (2.4)$$

with n_i being real numbers. The parameter Ω is ranging from 0 to 1. As we shall see below, the numbers $(n_1, n_2, n_3, n_4, n_5)$ correspond to critical values of the set of chemical potentials $(\omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3)$. Thus, as Ω is sent towards 1, we approach a critical value of the set of chemical potentials.

Employing (2.4), we can then write the partition function (2.3) as

$$Z_{\lambda,N}(\beta, \Omega) = \mathrm{Tr}_M\left[e^{-\beta D + \beta \Omega J}\right] = \mathrm{Tr}_M\left[e^{-\beta(D-J) - \beta(1-\Omega)J}\right] \qquad (2.5)$$

where we defined

$$J \equiv n_1 S_1 + n_2 S_2 + n_3 J_1 + n_4 J_2 + n_5 J_3. \qquad (2.6)$$

In general we can write the dilatation operator D for small λ as

$$D = D_0 + \lambda D_2 + \lambda^{3/2} D_3 + \lambda^2 D_4 + \cdots \qquad (2.7)$$

Here D_0 corresponds to the bare scaling dimension, D_2 the one-loop correction, and so on.

We now want to consider taking a limit with the temperature $T = 1/\beta$ going to zero. Focusing first on the free case $\lambda = 0$ the partition function is

$$Z_{\lambda=0,N}(\beta, \Omega) = \text{Tr}_M \left[e^{-\beta(D_0 - J) - \beta(1-\Omega)J} \right]. \tag{2.8}$$

For all the letters in $\mathcal{A}$ we have that D_0, S_1, S_2, J_1, J_2 and J_3 are integers or half-integers. Thus, given the numbers $(n_1, n_2, n_3, n_4, n_5)$, we can find a number $b > 0$ such that for any state with $D_0 - J \neq 0$ we have that $|D_0 - J| \geq b$. Therefore, for $\beta \to \infty$, all states with $D_0 - J > 0$ decouple from the partition function. We also see that if we have states with $D_0 - J < 0$ the partition function diverges. Thus, we restrict ourselves to choices of $(n_1, n_2, n_3, n_4, n_5)$ for which all states obey that $D_0 \geq J$. On the other hand, to avoid that all states decouple for $\beta \to \infty$ we see that we need to choose $(n_1, n_2, n_3, n_4, n_5)$ such that there are states with $D_0 = J$. Considering again (2.8) we see that to get a non-trivial partition function we need to keep $\beta(1-\Omega)$ fixed as $\beta \to \infty$. Thus, taking the limit

$$\beta \to \infty, \quad \tilde{\beta} \equiv \beta(1 - \Omega) \text{ fixed} \tag{2.9}$$

the partition function (2.8) becomes

$$Z_N(\tilde{\beta}) = \text{Tr}_{\mathcal{H}} \left[e^{-\tilde{\beta} D_0} \right] \tag{2.10}$$

where the trace is over the subset $\mathcal{H}$ of M given by

$$\mathcal{H} = \{ \alpha \in M | (D_0 - J)\alpha = 0 \}. \tag{2.11}$$

We see from (2.10) that it makes sense to interpret $\tilde{T} = 1/\tilde{\beta}$ as a temperature for the effective theory that one gets after taking the decoupling limit (2.9).

Considering now the case with non-zero coupling λ we see that for small λ the partition function (2.5) is

$$Z_{\lambda,N}(\beta, \Omega) = \text{Tr}_M \left[e^{-\beta(D_0 - J) - \beta\lambda D_2 - \beta(1-\Omega)J + \beta\mathcal{O}(\lambda^{3/2})} \right]. \tag{2.12}$$

Therefore, we get a non-trivial interaction term only if we keep $\beta\lambda$ fixed in the $\beta \to \infty$ limit.

We can now formulate the full decoupling limit. First, we assume that the numbers $(n_1, n_2, n_3, n_4, n_5)$ are given such that

- We have that $D_0 \geq J$ for any letter in $\mathcal{A}$.

- There exist letters in $\mathcal{A}$ for which $D_0 = J$.

These two conditions are equivalent to demanding that $D_0 \geq J$ for all states and that there exist states for which $D_0 = J$. With respect to $(n_1, n_2, n_3, n_4, n_5)$ we can then define the subset $\mathcal{H}$ of M as in (2.11). We now take the decoupling limit

$$\beta \to \infty, \quad \tilde{\beta} \equiv \beta(1 - \Omega) \text{ fixed}, \quad \tilde{\lambda} \equiv \frac{\lambda}{1 - \Omega} \text{ fixed}, \quad N \text{ fixed}. \tag{2.13}$$

	F_+	F_0	F_-	$\bar{F}_+$	$\bar{F}_0$	$\bar{F}_-$
SO(4)	$(1,-1)$	$(0,0)$	$(-1,1)$	$(1,1)$	$(0,0)$	$(-1,-1)$
SU(4)	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$

Table 1: Gauge field strength components in $\mathcal{N}=4$ SYM.

This brings us near to a point with zero temperature, $\Omega = 1$ and zero coupling. From the above considerations we see that the decoupling limit (2.13) of the full partition function of $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$ with gauge group SU(N) in the grand canonical ensemble becomes

$$Z_{\tilde{\lambda},N}(\tilde{\beta}) = \text{Tr}_{\mathcal{H}}\left[e^{-\tilde{\beta}(D_0 + \tilde{\lambda}D_2)}\right]. \tag{2.14}$$

This decoupled partition function can be thought of as a partition function for a decoupled theory, with the set of operators (and corresponding states) of the theory being $\mathcal{H}$, as defined in (2.11), with the effective temperature being $\tilde{T} = 1/\tilde{\beta}$ and with effective Hamiltonian being $D_0 + \tilde{\lambda}D_2$.

Several remarks are in order at this point:

- The higher loop terms in the dilatation operator $D_{n\geq 3}$ become negligible in the limit (2.13). Thus, the interaction truncates so that it only contains the one-loop contribution D_2.

- So far we have not assumed anything about N, thus the above decoupling limit also works for finite N. Therefore, the partition function (2.14) depends in general on the three parameters $\tilde{\lambda}$, N and $\tilde{\beta}$.

- Our requirements for the choice of $(n_1, n_2, n_3, n_4, n_5)$ mean that $(T, \Omega) = (0, 1)$ is a critical point, i.e. that $(T, \omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3) = (0, n_1, n_2, n_3, n_4, n_5)$ is a critical point. This is one of the reasons why the limit (2.13) yields an interesting decoupled theory.

2.2 Systematic exploration

We now examine systematically all the possible decoupling limits of the type (2.13). To do this, we first describe the set of letters $\mathcal{A}$ and then we proceed to consider which choices of $(n_1, n_2, n_3, n_4, n_5)$ lead to a decoupling limit.

Letters of $\mathcal{N}=4$ SYM. The set of letters $\mathcal{A}$ of $\mathcal{N}=4$ SYM consists of 6 independent gauge field strength components, 6 complex scalars and 16 complex fermions, plus the descendants of these that one gets by applying the 4 components of the covariant derivative. We describe in the following how the letters transform in multiplets of the SO(4) and SU(4) subgroups of $P\,\text{SU}(2,2|4)$. The gauge field strength components transform in the representations $[0,0,0]_{(1,0)}$ and $[0,0,0]_{(0,1)}$, where $[k,p,q]_{(j_1,j_2)}$ refers to the $[k,p,q]$ representation of SU(4) and the (j_1,j_2) representation of SU(2) $\times$ SU(2) = SO(4). The gauge field strength components have bare dimension $D_0 = 2$. We list the explicit weights for the gauge field strength components in table 1. The 6 complex scalars transform in the $[0,1,0]_{(0,0)}$ representation. They have bare scaling dimension $D_0 = 1$ and their weights are

	Z	X	W	$\bar{Z}$	$\bar{X}$	$\bar{W}$
SO(4)	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
SU(4)	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	$(-1,0,0)$	$(0,-1,0)$	$(0,0,-1)$

Table 2: Scalars of $\mathcal{N}=4$ SYM.

	$\chi_1, \chi_3, \chi_5, \chi_7$	$\chi_2, \chi_4, \chi_6, \chi_8$	$\bar{\chi}_1, \bar{\chi}_3, \bar{\chi}_5, \bar{\chi}_7$	$\bar{\chi}_2, \bar{\chi}_4, \bar{\chi}_6, \bar{\chi}_8$
SO(4)	$(\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(-\frac{1}{2}, -\frac{1}{2})$

Table 3: SO(4) weights for the fermions of $\mathcal{N}=4$ SYM.

	χ_1, χ_2	χ_3, χ_4	χ_5, χ_6	χ_7, χ_8
SU(4)	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$

Table 4: SU(4) weights for the $\chi_1, \ldots, \chi_8$ fermions of $\mathcal{N}=4$ SYM.

	$\bar{\chi}_1, \bar{\chi}_2$	$\bar{\chi}_3, \bar{\chi}_4$	$\bar{\chi}_5, \bar{\chi}_6$	$\bar{\chi}_7, \bar{\chi}_8$
SU(4)	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$

Table 5: SU(4) weights for the $\bar{\chi}_1, \ldots, \bar{\chi}_8$ fermions of $\mathcal{N}=4$ SYM.

listed in table 2. There are 16 complex fermion letters corresponding to the components of the complex fermionic fields χ_A^α, $\bar{\chi}_{\dot\alpha}^A$, $\alpha, \dot\alpha = 1, 2$, $A = 1, 2, 3, 4$. Half of the 16 complex fermions, denoted $\chi_1, \chi_2, \ldots, \chi_8$, transform in the $[0, 0, 1]_{(1/2,0)}$ representation and the other (conjugate) half, denoted $\bar{\chi}_1, \bar{\chi}_2, \ldots, \bar{\chi}_8$, transform in the $[1, 0, 0]_{(0,1/2)}$ representation. The fermions have bare scaling dimension $D_0 = \frac{3}{2}$. We have listed the SO(4) weights of the fermions in table 3. The SU(4) weights for $\chi_1, \chi_2, \ldots, \chi_8$ are listed in table 4 while the ones for $\bar{\chi}_1, \bar{\chi}_2, \ldots, \bar{\chi}_8$ are listed in table 5. Note that the both the SO(4) and the SU(4) representations are non-trivial for the fermions, contrary to the gauge field strength and the scalars. Finally there are the four components of the covariant derivative. They are not letters by themselves, but by combining any number of covariant derivations with a gauge field strength component, a scalar, or a complex fermion, one gets a letter in $\mathcal{A}$. The covariant derivative transforms in the representation $[0, 0, 0]_{(1/2,1/2)}$. The covariant derivative components contribute with $D_0 = 1$ to the bare scaling dimension of a letter. We have listed the weights in table 6.

In appendix A we review the oscillator representation for the letters $\mathcal{A}$ which gives an alternative way of representing $\mathcal{A}$.

Determination of the possible limits. From section 2.1 we have that a decoupling limit is defined by $n = (n_1, n_2, n_3, n_4, n_5)$. We now examine systematically what are the possible choices of n leading to a decoupling limit.

We begin by remarking that with respect to the bosons (the scalars, the gauge field strength components and the derivatives) we can choose $n_i \geq 0$ without loss of generality. This is not the case for the fermions, since the representation of SO(4) is linked to that of SU(4). However, if we allow for one of the n_i to be negative, we can choose the other four to be positive. We make the choice that n_1, n_3, n_4 and n_5 should be positive, or zero,

	d_1	d_2	$\bar{d}_1$	$\bar{d}_2$
SO(4)	$(1,0)$	$(0,1)$	$(-1,0)$	$(0,-1)$
SU(4)	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$

Table 6: Derivative operators of $\mathcal{N} = 4$ SYM.

whereas we allow n_2 to be negative. This is done without loss of generality.

As described in section 2.1 we have two constraints on $(n_1, n_2, n_3, n_4, n_5)$ in order to have a decoupling limit. The first constraint is that all letters should obey the inequality $D_0 \geq J$ and the second constraint is that there should be at least one letter for which $D_0 = J$.

Consider the first constraint. For the three scalars Z, X, W and the three derivatives d_1, d_2, $\bar{d}_2$ it implies the inequalities $n_1 \leq 1$, $-1 \leq n_2 \leq 1$, $n_3 \leq 1$, $n_4 \leq 1$ and $n_5 \leq 1$. We now impose the extra assumption that $n_3 \geq n_4 \geq n_5$ and $n_1 \geq n_2$, without loss of generality. We get therefore the following constraints on n_i

$$0 \leq n_1 \leq 1, \quad -1 \leq n_2 \leq n_1, \quad 0 \leq n_5 \leq n_4 \leq n_3 \leq 1. \tag{2.15}$$

We now turn to the fermions. It is evident that the number of $+1/2$ in the SO(4) and SU(4) weights is either 0, 2 or 4. From this, we see that only the fermions with four $+1/2$ in the SO(4) and SU(4) weights can give extra constraints on the n_i beyond (2.15).[2] The fermions with four $+1/2$ are χ_1, χ_2, $\bar{\chi}_3$, $\bar{\chi}_5$, $\bar{\chi}_7$. Of these five, only χ_1 and $\bar{\chi}_7$ are seen to give new constraints beyond (2.15). Thus, the constraints on the n_i that we get from the fermions are summarized into the single constraint

$$n_1 + n_3 + n_4 + |n_5 - n_2| \leq 3. \tag{2.16}$$

In conclusion we have, with the choices for the n_i made above, that the constraint that $D_0 \geq J$ for all letters in $\mathcal{A}$ is equivalent to the constraints (2.15) and (2.16) for the n_i.

We now turn to the second constraint on the n_i stating that there should be at least one letter in $\mathcal{A}$ such that $D_0 = J$. Concerning the complex scalars, it is clear that the number of scalars after the decoupling, i.e. with $D_0 = J$, is equal to how many of n_3, n_4 and n_5 are equal to 1. On the other hand, it is clear that the letters $\bar{Z}$, $\bar{X}$ and $\bar{W}$ can never be present. For the components of the covariant derivative we have similarly that the number of derivatives is equal to how many of n_1 and $|n_2|$ are equal to 1, and that the derivative operator $\bar{d}_1$ cannot be part of any decoupled theory. For the field strength components, we see that the only two possibilities for a field strength component surviving are if $n_1 = n_2 = 1$, giving $\bar{F}_+$, or if $n_1 = -n_2 = 1$, giving F_+.

For the fermions we see that we have one or more fermions if and only if $n_1 + n_3 + n_4 + |n_5 - n_2| = 3$. In particular, we get the fermion χ_1 if $n_1 - n_2 + n_3 + n_4 + n_5 = 3$ and the fermion $\bar{\chi}_7$ if $n_1 + n_2 + n_3 + n_4 - n_5 = 3$. Having more than one fermion is only

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[2]This is including the possibility of negative n_2.

# derivatives	0	0	0	0	1	1	1	1	2	2	2	2
# scalars	0	1	2	3	0	1	2	3	0	1	2	3
0 fermions		+	+			+			+			
1 fermion	+	+	+		+	+			+			
2 fermions				+			+			+		
5 fermions												+

Table 7: The fourteen possible decoupled theories.

possible in the following cases

$$
\begin{aligned}
n = (a, a, 1, 1, 1) \quad &: \chi_1, \chi_2 \\
n = (1, a, 1, 1, a) \quad &: \chi_1, \bar{\chi}_7 \\
n = (1, 1, 1, a, a) \quad &: \bar{\chi}_5, \bar{\chi}_7 \\
n = (0, -1, 1, 1, 0) \quad &: \chi_1, \bar{\chi}_8 \\
n = (1, -1, 1, 0, 0) \quad &: \chi_1, \chi_3 \\
n = (1, 1, 1, 1, 1) \quad &: \chi_1, \chi_2, \bar{\chi}_3, \bar{\chi}_5, \bar{\chi}_7
\end{aligned}
\tag{2.17}
$$

where $0 \leq a < 1$. Here we recorded which fermions are present in each case. We see that we can have either zero, one, two or five fermions surviving a decoupling limit.

We can now explore systematically what possible number of scalars, derivatives and fermions can be present in a decoupled theory after a decoupling limit. Note that there is precisely one gauge field strength component present if and only if we have two derivatives present. We begin by considering having 3 scalars and 2 derivatives present, i.e. the maximally possible number of scalars and derivatives. In this case, the only limit obeying (2.16) is $n = (1, 1, 1, 1, 1)$, and we see from (2.17) that this limit has five fermions present. Consider instead the case of having the number of scalars plus derivatives equal to four. Taking into account all the possibilities, it is easily seen that none of them can obey the constraint (2.16). If the number of scalars plus derivatives is equal to three it is not hard to see from the constraints (2.15) and (2.16) that the n_i can take five different forms, all of them listed in (2.17). Thus, all of these five possibilities lead to having precisely two fermions present. Finally, if the number of scalars plus derivatives is less than or equal to two all possibilities are realized, as one can see explicitly by our list of decoupling limits below in section 2.3. This is with the obvious exception of having zero scalars, fermions and derivatives, and having one derivative without any scalar or fermion. Altogether, we obtain 14 different decoupling limits, with the field content listed in table 7. We write explicit choices of the n_i for each of the 14 limits below in section 2.3.

2.3 List of decoupling limits

We list here the fourteen possible decoupling limits of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with gauge group $SU(N)$. The decoupling limits are all of the form (2.13) and they are specified by the numbers $(n_1, n_2, n_3, n_4, n_5)$. The fourteen limits give rise to fourteen different decoupled theories. For each decoupled theory we give the letter content and we state in which

representation of the symmetry algebra the letters transform. Note that the Dynkin labels of the algebras used in the following are explained in appendix B.[3]

The bosonic U(1) **limit.** Given by $n = (0, 0, 1, 0, 0)$. Letter content: Z. This limit has previously been considered in [6].

The fermionic U(1) **limit.** Given by $n = (\frac{3}{5}, -\frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5})$. Letter content: χ_1.

The SU(2) **limit.** Given by $n = (0, 0, 1, 1, 0)$. Letter content: Z, X. The letters transform in the [1] representation (i.e. spin $1/2$ representation) of $su(2)$. This limit has previously been considered in refs. [6–8].

The SU(1|1) **limit.** Given by $n = (\frac{2}{3}, 0, 1, \frac{2}{3}, \frac{2}{3})$. Letter content: Z, χ_1. The letters transform in the [1] representation of $su(1|1)$.

The SU(1|2) **limit.** Given by $n = (\frac{1}{2}, 0, 1, 1, \frac{1}{2})$. Letter content: Z, X and χ_1. The letters transform in the [1, 0] representation of $su(1|2)$.

The SU(2|3) **limit.** Given by $n = (0, 0, 1, 1, 1)$. Letter content: Z, X, W, χ_1 and χ_2. The letters transform in the [0, 0, 0, 1] representation of $su(2|3)$. This limit has previously been considered in refs. [6, 7].

The bosonic SU(1, 1) **limit.** Given by $n = (1, 0, 1, 0, 0)$. Letter content: $d_1^n Z$. The letters transform in the [−1] representation (i.e. spin $−1/2$ representation) of $su(1, 1)$.

The fermionic SU(1, 1) **limit.** Given by $n = (1, 0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. Letter content: $d_1^n \chi_1$. The letters transform in the [−2] representation (i.e. spin $−1$ representation) of $su(1, 1)$.

The SU(1, 1|1) **limit.** Given by $n = (1, 0, 1, \frac{1}{2}, \frac{1}{2})$. Letter content: $d_1^n Z$ and $d_1^n \chi_1$. The letters transform in the [0, 1] representation of $su(1, 1|1)$.

The SU(1, 1|2) **limit.** Given by $n = (1, 0, 1, 1, 0)$. Letter content: $d_1^n Z$, $d_1^n X$, $d_1^n \chi_1$ and $d_1^n \bar{\chi}_7$. The letters transform in the [0, 1, 0] representation of $su(1, 1|2)$.

The SU(1, 2) **limit.** Given by $n = (1, 1, 0, 0, 0)$. Letter content: $d_1^n d_2^k \bar{F}_+$. The letters transform in the [0, −3] representation of $su(1, 2)$.

The SU(1, 2|1) **limit.** Given by $n = (1, 1, \frac{1}{2}, \frac{1}{2}, 0)$. Letter content: $d_1^n d_2^k \bar{F}_+$, $d_1^n d_2^k \bar{\chi}_7$. The letters transform in the [0, 0, 2] representation of $su(1, 2|1)$.

The SU(1, 2|2) **limit.** Given by $n = (1, 1, 1, 0, 0)$. Letter content: $d_1^n d_2^k \bar{F}_+$, $d_1^n d_2^k Z$, $d_1^n d_2^k \bar{\chi}_5$, $d_1^n d_2^k \bar{\chi}_7$. The letters transform in the [0, 0, 0, 1] representation of $su(1, 2|2)$.

The SU(1, 2|3) **limit.** Given by $n = (1, 1, 1, 1, 1)$. Letter content: $d_1^n d_2^k \bar{F}_+$, $d_1^n d_2^k Z$, $d_1^n d_2^k X$, $d_1^n d_2^k W$, $d_1^n d_2^k \chi_1$ and $d_1^n d_2^k \chi_2$, $d_1^n d_2^k \bar{\chi}_3$, $d_1^n d_2^k \bar{\chi}_5$, $d_1^n d_2^k \bar{\chi}_7$. The letters transform in the [0, 0, 0, 1, 0] representation of $su(1, 2|3)$.

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[3]See appendix A and appendix B for more details on the algebras and representations used in this section.

As explained in section 2.2, the above fourteen limits constitute a complete list of decoupling limits of the form (2.13). There are other possible choices of $(n_1, n_2, n_3, n_4, n_5)$ that give decoupling limits but the resulting theories are all equivalent to one of the theories listed above. For example, the limit given by $n = (\frac{1}{2}, \frac{1}{2}, 1, 1, 0)$ gives a decoupled theory containing Z, X and $\bar{\chi}_7$ but this theory is in fact equivalent to the SU(1|2) theory described above. A few of the decoupled theories can even be obtained from a continuous family of choices for n. The fermionic U(1) theory can for instance be found from $n = (a, -a, b, b, b)$ with $0 < a, b < 1$ satisfying $2a + 3b = 3$.

The above list of decoupling limits can be divided into the two trivial limits, being the bosonic and fermionic U(1) limits, and the twelve non-trivial limits. The twelve non-trivial decoupled theories can be divided into groups according to the effective dimensionality of the decoupled theory. The SU(2), SU(1|1), SU(1|2) and SU(2|3) theories are effectively zero-dimensional so they correspond to Quantum Mechanical theories. Two of these were found in [6]. The bosonic SU(1, 1), fermionic SU(1, 1), SU(1, 1|1) and SU(1, 1|2) theories all have one derivative present, thus they are effectively one-dimensional. Finally, the SU(1, 2), SU(1, 2|1), SU(1, 2|2) and SU(1, 2|3) theories are effectively two-dimensional, since they each have two derivatives present.

It is important to note that the above list of limits and theories are in good correspondence with the list of consistent subgroups of the $P\,\mathrm{SU}(2, 2|4)$ symmetry of $\mathcal{N} = 4$ SYM at the one-loop order, as examined in [12, 11]. The only exception is the so-called excitation sector for which the number of excitations is kept fixed, thus it is not in accordance with our decoupling limit (2.13).

2.4 Closure of D_2 in the decoupling limits

We found above that in the decoupling limit (2.13) for a given $n = (n_1, n_2, n_3, n_4, n_5)$ only the states with $D_0 = J$ survive and the effective Hamiltonian for the theory becomes $D_0 + \tilde{\lambda} D_2$. In the following we show that this is consistent with the D_2 operator.

We begin by reviewing briefly the D_2 operator, as found by Beisert [12]. The D_2 operator acts on two letters at a time in a given operator. We can therefore think of D_2 in terms of the action on $\mathcal{A} \times \mathcal{A}$, i.e. on the product of two singleton representations of $psu(2, 2|4)$. It is found that $\mathcal{A} \times \mathcal{A}$ splits up in a sum of representations as follows

$$\mathcal{A} \times \mathcal{A} = \sum_{j=0}^{\infty} \mathcal{V}_j \tag{2.18}$$

where the singleton representation $\mathcal{A}$ and the modules $\mathcal{V}_j$ are [13, 14]

$$\mathcal{A} = \mathcal{B}_{[0,1,0]_{(0,0)}}^{\frac{1}{2}, \frac{1}{2}} \qquad \mathcal{V}_0 = \mathcal{B}_{[0,2,0]_{(0,0)}}^{\frac{1}{2}, \frac{1}{2}} \qquad \mathcal{V}_1 = \mathcal{B}_{[1,0,1]_{(0,0)}}^{\frac{1}{4}, \frac{1}{4}} \qquad \mathcal{V}_j = \mathcal{C}_{[0,0,0]_{(\frac{j}{2}-1, \frac{j}{2}-1)}}^{1,1} \quad \text{for } j \geq 2 \tag{2.19}$$

written in the notation of [13], where for each module it is specified which superconformal primary operator the representation is generated from. With this we can write the D_2 operator as [12]

$$D_2 = -\frac{1}{2N} \sum_{j=0}^{\infty} h(j)(P_j)_{CD}^{AB} : \mathrm{Tr}[W_A, \bar{W}^C][W_B, \bar{W}^D] : \tag{2.20}$$

where $h(j) = \sum_{k=1}^{j} \frac{1}{k}$ are the harmonic numbers, P_j is the projection operator to the module $\mathcal{V}_j$ and W_A represent all possible letters of $\mathcal{N} = 4$ SYM.

The D_2 operator (2.20) commutes by construction with all the generators of the tree-level superconformal algebra $psu(2, 2|4)$ (see appendix A) [12]. In particular, this means that

$$[D_2, D_0] = 0, \quad [D_2, S_a] = 0, \quad [D_2, J_i] = 0 \tag{2.21}$$

with $a = 1, 2$ and $i = 1, 2, 3$. As a consequence of this, we see that

$$[D_2, D_0 - J] = 0 \tag{2.22}$$

with J as defined in (2.6).

Using eq. (2.22) we can now show that D_2 is closed in any of the decoupled theories listed in section 2.3. For a decoupling limit with a given $n = (n_1, n_2, n_3, n_4, n_5)$ the states in the corresponding decoupled theory are the ones with $D_0 - J = 0$. Therefore, eq. (2.22) means that the decoupled theory is closed with respect to D_2 since the action of D_2 on any state with $D_0 - J = 0$ will give a new state with $D_0 - J = 0$.

3. Spectrum of decoupled theories in planar limit

In this section we consider the decoupled theories found in section 2 in the planar limit. In the planar limit it is possible to single out the single-trace operators, and the spectrum of the multi-trace operators can be found from the knowledge of the spectrum of the single-trace operators. Using furthermore the spin chain interpretation for the single-trace operators [5] it is possible to find a Bethe equation that contains the full spectrum of the effective Hamiltonian $D_0 + \tilde{\lambda} D_2$. We review how this works, and we use this to obtain explicitly the low energy spectrum for the decoupled theories found in section 2 in the planar limit.

Note that the technology used to find the spectrum of D_2 has been developed mainly in [5, 15, 12, 16]. In this section we apply this technology to derive the specific spectra for the effective Hamiltonian $D_0 + \tilde{\lambda} D_2$.

3.1 Full spectrum from Bethe equations

In the planar limit of $\mathcal{N} = 4$ SYM, a single-trace operator with L letters

$$\text{Tr}(A_1 A_2 \cdots A_L) \tag{3.1}$$

can be interpreted as a state of a periodic homogenous spin chain of length L where each letter in the trace corresponds to a spin in one site of the spin chain [5]. The simplest example of this correspondence is the SU(2) sector which contains only two types of letters, Z and X, corresponding to the spin-up and spin-down states in the spin chain. The dynamics of the spin chain is governed by a Hamiltonian which in our case is $D_0 + \tilde{\lambda} D_2$. The spectrum of D_2 for $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the planar limit is given by the $P\,\text{SU}(2, 2|4)$ super spin chain found in [16]. In the following we use this to find the spectrum of D_2 in the planar limit for the various decoupled theories.

For the decoupled theories that contain one or more of the complex scalars Z, X and W, the vacuum sector consists of the symmetrized combinations of the scalars, e.g. for the SU(2) theory the vacuum states are of the form $\text{Tr}(\text{sym}(Z^m X^n))$. The value of D_2 on such states is zero, which is connected to the fact that these particular single-trace operators correspond to chiral primaries in $\mathcal{N} = 4$ SYM. See [7] for a discussion of this for the SU(2) theory.

There are three decoupled theories which do not contain any of the scalars Z, X and W, and for these the D_2 vacuum energy is shifted from zero. The fermionic SU(1, 1) theory has ground state $\text{Tr}(\chi_1^L)$ with D_2 eigenvalue L, the SU(1, 2) theory has ground state $\text{Tr}(\bar{F}_+^L)$ with D_2 eigenvalue $3L/2$ and the SU(1, 2|1) theory has ground state $\text{Tr}(\bar{\chi}_7^L)$ with D_2 eigenvalue L. As we explain below and in section 4.5, this has important implications for considering the large $\tilde{\lambda}$ limit.

The effective Hamiltonian for our decoupled theories is

$$H = D_0 + \tilde{\lambda}D_2. \tag{3.2}$$

The Bethe ansatz technique is only relevant for the D_2 part of the Hamiltonian. Instead, for the D_0 part we use that any eigenstate of the spin chain is an eigenstate of D_0. In general the D_0 eigenvalue will depend on the excitations of the spin chain. This dependence can in many cases be interpreted as a Zeeman coupling to an external magnetic field, as we shall see below in section 3.2.

The spin chains that correspond to the planar limit of the decoupled theories found in section 2 are all integrable and the spectrum of D_2 for each of them is determined by using the Bethe ansatz technique [11]. By using the Dynkin labels V_a and Cartan matrix M_{ab} of each decoupled theory (see appendix B), we can treat them at the same time and obtain the spectrum of D_2 from the generalized Bethe equation. Each eigenstate of D_2 is determined by a set of Bethe roots u_k, $k = 1, \ldots, K$, where K is the total number of excitations. Some of our decoupled theories have a symmetry algebra of rank higher than one and for these theories it is important to specify which simple root of the Dynkin diagram each Bethe excitation corresponds to. This is done with the label j_k which for each Bethe excitation can take values from one and up to the rank of the symmetry algebra.

The eigenvalue of D_2 on a state with K excitations is given by [16]

$$D_2 = \frac{1}{2} \sum_{k=1}^{K} \frac{|V_{j_k}|}{u_k^2 + \frac{1}{4}V_{j_k}^2} + cL \tag{3.3}$$

where we have included the possible shift cL with $c \in \{0, 1, 3/2\}$, depending on the ground state of the theory as discussed above. It turns out that our decoupled theories all have the property that only one of the Dynkin labels is non-vanishing. Therefore only excitations corresponding to this one non-zero Dynkin label will give contributions to the spectrum.

The Bethe roots are determined by the general Bethe equations that can be written in compact form as [16, 11]

$$\left(\frac{u_k + \frac{i}{2}V_{j_k}}{u_k - \frac{i}{2}V_{j_k}}\right)^L = \prod_{\ell=1, \ell \neq k}^{K} \left(\frac{u_k - u_\ell + \frac{i}{2}M_{j_k, j_\ell}}{u_k - u_\ell - \frac{i}{2}M_{j_k, j_\ell}}\right) \tag{3.4}$$

with the cyclicity condition

$$U = \prod_{k=1}^{K} \left(\frac{u_k + \frac{i}{2} V_{j_k}}{u_k - \frac{i}{2} V_{j_k}} \right) \tag{3.5}$$

where $U = 1$ for the decoupled theories with bosonic vacua and $U = (-1)^L$ for the two decoupled theories in which we have a fermionic vacuum state. The full spectrum of the effective Hamiltonian (3.2) in the planar limit is determined by eqs. (3.3)–(3.5) for all decoupled theories.

Some of the decoupled theories considered here are well known in the Condensed Matter literature. The SU(2) theory is for example equivalent to the Heisenberg $XXX_{1/2}$ model while the bosonic SU(1,1) theory is the non-compact $XXX_{-1/2}$ Heisenberg model and the fermionic SU(1,1) is the non-compact spin -1 XXX model [11]. The SU(1|1) theory is equivalent to a Heisenberg $XX_{1/2}$ spin chain in an external magnetic field which describes free fermions and is exactly solvable. We will discuss this theory further in section 4.4. Finally, the SU(1|2) theory is equivalent to the so called $t - J$ model [17] that is believed to be relevant for high T_c superconductivity.

We see thus, that our decoupling limits (2.13) for planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ lead to known Condensed Matter theories which are fully decoupled. In other words, when approaching certain of the critical points found in section 2, planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ reduces to known Condensed Matter theories.

3.2 Low energy spectrum in the thermodynamic limit

It is in general hard to solve explicitly the Bethe equations (3.4)–(3.4), but we can easily obtain a leading order solution for the low energy spectrum in the thermodynamic limit $L \to \infty$. In this regime the positions of the roots u_k scale like L [5] and we therefore define $u_k = L\tilde{u}_k$. Plugging this into eq. (3.4) and taking the logarithm, we find

$$2\pi n_k - \frac{V_{j_k}}{\tilde{u}_k} = \frac{1}{L} \sum_{\ell=1, \ell \neq k}^{K} \frac{M_{j_k, j_\ell}}{\tilde{u}_\ell - \tilde{u}_k} + \mathcal{O}(L^{-2}), \tag{3.6}$$

where n_k are integers. Neglecting the right hand side to leading order in $1/L$, eq. (3.6) gives the solution

$$\tilde{u}_k = \frac{V_{j_k}}{2\pi n_k} + \mathcal{O}(L^{-1}) \tag{3.7}$$

and inserting that into the spectrum (3.3) we obtain

$$D_2 = \frac{2\pi^2}{L^2} \sum_{k=1}^{K'} \frac{n_k^2}{|V_{j_k}|} + \mathcal{O}(L^{-3}) \tag{3.8}$$

where the sum now only goes over the Bethe roots that correspond to the simple root of the Dynkin diagram with non-vanishing Dynkin label.

SU($\cdot$)	(2)	$(1,1)_{\text{bos}}$	(1|1)	(1|2)	(2|3)	(1,1|1)	(1,1|2)	(1,2|2)	(1,2|3)
a	1	0	0	1	2	0	1	0	2
b	0	1	0	0	0	1	1	2	2
c	0	0	1	1	2	1	2	2	4

Table 8: The table shows how many number operators we have of each type (a for scalars M_n, b for derivatives N_n, and c for fermions F_n) in each of the nine theories that contain at least one scalar. $\text{SU}(1,1)_{\text{bos}}$ corresponds to the bosonic SU(1,1) theory. The numbers in this table will be used again in section 4.3 where we consider the partition functions for large $\tilde{\lambda}$.

Plugging the leading order solution (3.7) into the constraint equation (3.5) gives

$$\sum_{k=1}^{K'} n_k = 0 \qquad (3.9)$$

which is the zero-momentum condition for the spin chain and the cyclicity condition for the trace on the gauge theory side. For bosonic excitations we can have more than one excitation with the same n_k, whereas for fermionic excitations we can at most have one excitation with a given value of n_k. We must therefore distinguish between scalar excitations, derivatives and fermionic excitations. For the two possible scalar excitations, we denote the number of n_k that are equal to a particular integer n as $M_n^{(i)}$, $i = 1, 2$, for the two derivative excitations we denote the number as $N_n^{(j)}$, $j = 1, 2$, and for the four possible fermionic excitations as $F_n^{(\alpha)}$, $\alpha = 1, \ldots, 4$. From the oscillator representation in appendix B we can see that not all excitations are independent, $\bar{F}_+$ is for example a composite field and we do not need to keep track of it in the partition function. The same is true for $\bar{\chi}_3$ which is composed of the $\bar{\chi}_7$ and W excitations. Therefore we only have four different types of fermionic excitations and not five as one might guess in a theory with five fermions like the SU(1,2\|3).

All decoupled theories containing scalars. Nine out of the 12 non-trivial decoupled theories found in section 2 contain at least one scalar and their spectra can all be described in the same way using the number operators M_n, N_n and F_n. Depending on their letter content, the decoupled theories have different number of these operators appearing and in table 8 we list how many there are of each of the three possible types. These theories all share the feature that the absolute value of the single non-zero Dynkin label is equal to one and therefore the spectra for these nine different theories all take the form

$$H = L + \sum_{n\in\mathbb{Z}} \left(\sum_{j=1}^{b} N_n^{(j)} + \frac{1}{2} \sum_{\alpha=1}^{c} F_n^{(\alpha)} \right) + \frac{2\pi^2 \tilde{\lambda}}{L^2} \sum_{n\in\mathbb{Z}} n^2 \left(\sum_{i=1}^{a} M_n^{(i)} + \sum_{j=1}^{b} N_n^{(j)} + \sum_{\alpha=1}^{c} F_n^{(\alpha)} \right) \qquad (3.10)$$

with the cyclicity (zero momentum) constraint

$$P \equiv \sum_{n\in\mathbb{Z}} n \left(\sum_{i=1}^{a} M_n^{(i)} + \sum_{j=1}^{b} N_n^{(j)} + \sum_{\alpha=1}^{c} F_n^{(\alpha)} \right) = 0. \qquad (3.11)$$

Note that $F_n^{(\alpha)} \in \{0, 1\}$ while $M_n^{(i)}, N_n^{(j)} \in \{0, 1, 2, \ldots\}$. The numbers a, b and c are given in table 8.

The first two terms in the spectrum come from D_0. Recall that in the decoupling limit we have that $D_0 = J$. The vacuum is made from the scalars which all contribute 1 to J and therefore the vacuum has $J = L$. Each derivative gives an additional contribution and $\sum_{n \in \mathbb{Z}} \sum_{j=1}^{b} N_n^{(j)}$ precisely counts the number of derivatives. Similarly $\sum_{n \in \mathbb{Z}} \sum_{\alpha=1}^{c} F_n^{(\alpha)}$ counts the total number of fermions and each of them contributes $1/2$ more to J than the scalars do. The second term can be interpreted as a coupling of the spin chain to an external magnetic field through a Zeeman term [8].

Decoupled theories without scalars. The three decoupled theories without scalars have their D_2 vacuum shifted from zero. We can still use the Bethe ansatz to find their low energy spectrum, but it will not be as useful to us when we consider the large $\tilde{\lambda}$ Hagedorn temperature.

The fermionic $SU(1, 1)$ theory is the simplest example of a decoupled theory with a non-vanishing D_2 vacuum energy. The ground state is made from fermions and we assume that L is odd to satisfy the cyclicity constraint. Since χ_1 is now the highest weight, the representation has Dynkin label $V = -2$ and this theory is equivalent to the Heisenberg XXX_{-1} spin chain [11]. The spectrum of the Hamiltonian is

$$H = \left(\frac{3}{2} + \tilde{\lambda} \right) L + \sum_{n \in \mathbb{Z}} N_n^{(1)} + \frac{\pi \tilde{\lambda}}{L^2} \sum_{n \in \mathbb{Z}} n^2 N_n^{(1)} \qquad (3.12)$$

where we note that $\tilde{\lambda}$ already appears in the first term. We also have the usual zero-momentum constraint analogous to eq. (3.11).

The $SU(1, 2)$ theory is very interesting since it shares many features with QCD, as discussed in section 6. The highest weight is $\bar{F}_+$ and the representation has Dynkin label $V = [0, -3]$. The spectrum of the Hamiltonian is

$$H = \left(2 + \frac{3}{2} \tilde{\lambda} \right) L + \sum_{n \in \mathbb{Z}} \left(N_n^{(1)} + N_n^{(2)} \right) + \frac{2\pi \tilde{\lambda}}{3L^2} \sum_{n \in \mathbb{Z}} n^2 \left(N_n^{(1)} + N_n^{(2)} \right). \qquad (3.13)$$

The third theory that does not contain any scalar is $SU(1, 2|1)$ where the highest weight is again χ_1 and the spectrum can straightforwardly be worked out along similar lines as for the fermionic $SU(1, 1)$ and the $SU(1, 2)$ decoupled theories.

The $\tilde{\lambda} L$ term that appears in the spectrum of these decoupled theories has important implications when $\tilde{\lambda}$ and L are large. This will be discussed in sections 4.5 and 5.2.

4. Finite temperature behavior in planar limit

In this section we begin by generalizing the computation of the partition function for free $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ [18 – 21, 6] to include all five possible chemical potentials. Applying then the decoupling limit (2.13) we find the partition function and Hagedorn temperature for each of the decoupled theories at zero coupling.

Turning on a small 't Hooft coupling λ, we compute in section 4.2 the one-loop correction to the Hagedorn temperature for small values of $\tilde{\lambda}$ in all the decoupled theories. The procedure is a generalization of the one used in [22, 6].

In section 4.3 we compute the Hagedorn temperature in the large $\tilde{\lambda}$ regime. This is done by using a general relation between the Hagedorn temperature in the decoupled theories and the free energy per site of the corresponding spin chain model in the thermodynamic limit [7]. We use this method for all the decoupled theories containing scalars.

In section 4.4 we examine the $SU(1|1)$ theory. We compute the one-loop Hagedorn temperature for all values of $\tilde{\lambda}$ using the relation with the Heisenberg $XX_{1/2}$ spin chain.

Finally in section 4.5 we study the large $\tilde{\lambda}$ regime of the $SU(1,2)$ theory which is particularly interesting for its connection to pure Yang-Mills theory.

4.1 Partition function of the free theory

In this section we compute of the partition function for free $\mathcal{N} = 4$ SU(N) SYM on $\mathbb{R} \times S^3$ with all five possible chemical potentials turned on. The partition function of free SU(N) SYM on $\mathbb{R} \times S^3$ can be found from the letter partition function [18–20]. With chemical potentials turned on, the only difference is that one needs the letter partition function with chemical potentials [20, 21, 6]. Below we compute the general letter partition function $z(x, \rho_j, y_i)$ depending on the temperature and all five chemical potentials, where we introduce the notation

$$x \equiv e^{-\beta}, \quad \rho_j \equiv e^{\beta\omega_j}, j = 1, 2, \quad y_i \equiv e^{\beta\Omega_i}, i = 1, 2, 3 . \tag{4.1}$$

With the letter partition function $z(x, \rho_j, y_i)$ one can then find the full partition function for free SU(N) $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ as

$$Z_{\lambda=0,N}(x, \rho_j, y_i) = \int [dU] \exp\left[\sum_{k=1}^{\infty} \frac{1}{k} z(\eta^{k+1} x^k, \rho_j^k, y_i^k) \left(\text{Tr}(U^k)\text{Tr}(U^{-k}) - 1\right)\right] \tag{4.2}$$

where $\eta = e^{2\pi i}$ is introduced to take the correct sign into account for the fermions. In the planar limit $N = \infty$, the partition function (4.2) becomes

$$\log Z_{\lambda=0,N=\infty}(x, \rho_j, y_i) = -\sum_{k=1}^{\infty} \log\left[1 - z(\eta^{k+1} x^k, \rho_j^k, y_i^k)\right] . \tag{4.3}$$

One can see from (4.3) that one encounters a singularity when $z(x, \rho_j, y_i) = 1$. This is the Hagedorn singularity for planar $\mathcal{N} = 4$ SU(N) SYM on $\mathbb{R} \times S^3$ [18–21, 6]. With chemical potentials, we see that the equation $z(x, \rho_j, y_i) = 1$ defines the Hagedorn temperature $T_H(\omega_j, \Omega_i)$ as a function of all the five chemical potentials.[4]

---

[4]Note that this means that (4.3) is only valid for temperatures below the Hagedorn temperature $T_H(\omega_j, \Omega_i)$. If we want to study the theory above the Hagedorn temperature we need to go beyond the planar limit. This is in accordance with the fact that the Hagedorn temperature $T_H(\omega_j, \Omega_i)$ is limiting for free $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the planar limit $N = \infty$, thus it takes an infinite amount of energy to reach the Hagedorn temperature.

In the following we first compute the full letter partition function of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with all five chemical potentials turned on. Then we take the decoupling limit (2.13) of the obtained letter partition function of $\mathcal{N} = 4$ SYM, thus finding the partition function for each of the decoupled theories in the free limit.[5] Employing the letter partition function for the decoupled theories we furthermore compute the Hagedorn temperature for each decoupled sector in the free limit. Note that the results of the computations for the decoupled theories are listed at the end of section 4.2.

Computing the letter partition function. We compute now the letter partition function for $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the presence of non-zero chemical potentials for the R-charges of the SU(4) R-symmetry and for the Cartan generators of the SO(4) symmetry group of S^3. To compute the letter partition function we use the spherical harmonic expansion method by expanding each field in the spectrum of $\mathcal{N} = 4$ SYM in terms of the corresponding spherical harmonics. To do this, instead of the Cartan generators of the SO(4) symmetry S_1 and S_2, it is convenient to define the operators

$$S_L = \frac{S_1 - S_2}{2}, \quad S_R = \frac{S_1 + S_2}{2} \tag{4.4}$$

corresponding to the generators of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry.

We begin with the scalars. The spherical harmonics corresponding to scalars are denoted by $S_{j,m,\bar{m}}(\alpha)$, where α represents the coordinates of S^3 and $m, \bar{m}$ label the eigenvalues of S_L and S_R respectively. Their values are $m = \bar{m} = -j/2, -j/2 + 1, \ldots, j/2 - 1, j/2$.

From table 2, we see that all the six scalars are in the same representation of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The scalar partition function can therefore be written as

$$\eta_S(x, \rho, \bar{\rho}, y_i) = \sum_{i=1}^{3} \sum_{j=0}^{\infty} \sum_{m=-j/2}^{j/2} \sum_{\bar{m}=-j/2}^{j/2} x^{j+1} \rho^m \bar{\rho}^{\bar{m}} (y_i + y_i^{-1}) \tag{4.5}$$

where we introduced the notation

$$\rho \equiv e^{\beta(\omega_1 - \omega_2)}, \quad \bar{\rho} \equiv e^{\beta(\omega_1 + \omega_2)}. \tag{4.6}$$

Performing the sums, we get the following result for the scalar partition function

$$\eta_S(x, \omega_j, y_i) = \frac{(x - x^3)}{(1 - xe^{\beta\omega_1})(1 - xe^{-\beta\omega_1})(1 - xe^{\beta\omega_2})(1 - xe^{-\beta\omega_2})} \sum_{i=1}^{3} \left(y_i + y_i^{-1} \right). \tag{4.7}$$

Turning to the vectors, we have that they are neutral under the R-charges. The spherical harmonics corresponding to the gauge boson in the representation $[0, 0, 0]_{(1,0)}$ are denoted by $V^L_{j,m,\bar{m}}(\alpha)$ with $m = -(j+1)/2, \ldots, (j+1)/2$ and $\bar{m} = -(j-1)/2, \ldots, (j-1)/2$. Their contribution to the letter partition function is given by

$$\eta_{V^L}(x, \rho, \bar{\rho}, y_i) = \sum_{j=1}^{\infty} \sum_{m=-(j+1)/2}^{(j+1)/2} \sum_{\bar{m}=-(j-1)/2}^{(j-1)/2} x^{j+1} \rho^m \bar{\rho}^{\bar{m}}. \tag{4.8}$$

---

[5]See also [23, 24].

The spherical harmonics corresponding to the gauge boson in the representation $[0,0,0]_{(0,1)}$ are denoted by $V_{j,m,\bar{m}}^R(\alpha)$ with $m = -(j-1)/2, \ldots, (j-1)/2$ and $\bar{m} = -(j+1)/2, \ldots, (j+1)/2$. Their contribution to the letter partition function is given by

$$\eta_{V^R}(x, \rho, \bar{\rho}, y_i) = \sum_{j=1}^{\infty} \sum_{m=-(j-1)/2}^{(j-1)/2} \sum_{\bar{m}=-(j+1)/2}^{(j+1)/2} x^{j+1} \rho^m \bar{\rho}^{\bar{m}}. \tag{4.9}$$

Performing the sums and adding together the two contributions, we get the following result for the vector partition function

$$\eta_V(x, \omega_j) = \frac{2x^2 \left[1 + 2\cosh(\beta\omega_1)\cosh(\beta\omega_2) - 2x\left(\cosh(\beta\omega_1) + \cosh(\beta\omega_2)\right) + x^2\right]}{(1 - xe^{\beta\omega_1})(1 - xe^{-\beta\omega_1})(1 - xe^{\beta\omega_2})(1 - xe^{-\beta\omega_2})}. \tag{4.10}$$

Finally, we turn to the fermions. Fermions appear in two representations, $[0,0,1]_{(1/2,0)}$ and $[1,0,0]_{(0,1/2)}$. For the representation $[0,0,1]_{(1/2,0)}$ we can introduce the spherical harmonics $F_{j,m,\bar{m}}^1(\alpha)$ with $m = -(j)/2, \ldots, (j)/2$ and $\bar{m} = -(j-1)/2, \ldots, (j-1)/2$. Taking into account the dependence on the R-charge chemical potentials for fermions in this representation which is given by

$$Y_1 = (y_1 y_2 y_3)^{1/2} + y_1^{1/2}(y_2 y_3)^{-1/2} + (y_1 y_3)^{-1/2} y_2^{1/2} + (y_1 y_2)^{-1/2} y_3^{1/2} \tag{4.11}$$

we obtain the letter partition function

$$\eta_{F^1}(x, \rho, \bar{\rho}, y_i) = Y_1 \sum_{j=1}^{\infty} \sum_{m=-(j)/2}^{(j)/2} \sum_{\bar{m}=-(j-1)/2}^{(j-1)/2} x^{j+\frac{1}{2}} \rho^m \bar{\rho}^{\bar{m}}. \tag{4.12}$$

The result in terms of ω_1 and ω_2 is given by

$$\eta_{F^1}(x, \omega_j, y_i) = Y_1 \frac{2x^{3/2} \left(\cosh\left[\beta\left(\frac{\omega_1 - \omega_2}{2}\right)\right] - x\cosh\left[\beta\left(\frac{\omega_1 + \omega_2}{2}\right)\right]\right)}{(1 - xe^{\beta\omega_1})(1 - xe^{-\beta\omega_1})(1 - xe^{\beta\omega_2})(1 - xe^{-\beta\omega_2})}. \tag{4.13}$$

For fermions in the representation $[1,0,0]_{(0,1/2)}$, we can introduce the spherical harmonics $F_{j,m,\bar{m}}^2(\alpha)$ with $m = -(j-1)/2, \ldots, (j-1)/2$ and $\bar{m} = -(j)/2, \ldots, (j)/2$. The dependence on the R-charge chemical potentials in this case is given by

$$Y_2 = (y_1 y_2 y_3)^{-1/2} + y_1^{-1/2}(y_2 y_3)^{1/2} + (y_1 y_3)^{1/2} y_2^{-1/2} + (y_1 y_2)^{1/2} y_3^{-1/2} \tag{4.14}$$

and the contribution to the letter partition function is

$$\eta_{F^2}(x, \rho, \bar{\rho}, y_i) = Y_2 \sum_{j=1}^{\infty} \sum_{m=-(j-1)/2}^{(j-1)/2} \sum_{\bar{m}=-(j)/2}^{(j)/2} x^{j+\frac{1}{2}} \rho^m \bar{\rho}^{\bar{m}}. \tag{4.15}$$

The result in terms of ω_1 and ω_2 is given by

$$\eta_{F^2}(x, \omega_j, y_i) = Y_2 \frac{2x^{3/2} \left(\cosh\left[\beta\left(\frac{\omega_1 + \omega_2}{2}\right)\right] - x\cosh\left[\beta\left(\frac{\omega_1 - \omega_2}{2}\right)\right]\right)}{(1 - xe^{\beta\omega_1})(1 - xe^{-\beta\omega_1})(1 - xe^{\beta\omega_2})(1 - xe^{-\beta\omega_2})}. \tag{4.16}$$

Adding together the contributions of scalars, vectors and fermions, we obtain the letter partition function for $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the presence of non-zero chemical potentials for the R-charges of the SU(4) R-symmetry and for the Cartan generators of the SO(4) symmetry group of S^3 which is given by

$$
\begin{aligned}
z\left(x, \omega_j, y_i\right) = \prod_{k=1}^{2} & \left((1 - xe^{\beta\omega_k})(1 - xe^{-\beta\omega_k})\right)^{-1} \left\{ (x - x^3) \sum_{l=1}^{3}(y_l + y_l^{-1}) \right. \\
& + 2x^2 \left[1 + 2\cosh(\beta\omega_1)\cosh(\beta\omega_2) - 2x\left(\cosh(\beta\omega_1) + \cosh(\beta\omega_2)\right) + x^2\right] \\
& + Y_1\, 2x^{3/2} \left[\cosh\left[\beta\left(\frac{\omega_1 - \omega_2}{2}\right)\right] - x\cosh\left[\beta\left(\frac{\omega_1 + \omega_2}{2}\right)\right]\right] \\
& \left. + Y_2\, 2x^{3/2} \left[\cosh\left[\beta\left(\frac{\omega_1 + \omega_2}{2}\right)\right] - x\cosh\left[\beta\left(\frac{\omega_1 - \omega_2}{2}\right)\right]\right] \right\}. \quad (4.17)
\end{aligned}
$$

As shown in appendix C, the above result for the letter partition function can also be obtained using the oscillator representation of $\mathcal{N} = 4$ SYM [25, 12].

With the letter partition function (4.17) in hand, the partition function of free $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with all five possible chemical potentials turned on is given by eq. (4.2), or eq. (4.3) in the planar limit.

Free partition functions for the decoupled theories. We can now find the partition function for each of the decoupled theories when $\tilde{\lambda} = 0$ by taking the decoupling limit (2.13). This is done by taking the decoupling limit of the letter partition function (4.17). Defining $Y \equiv \exp(i\beta\Omega)$, we can write the decoupling limit of the letter partition function as

$$
x \to 0, \; Y \to \infty \quad \text{with } \tilde{x} = xY \text{ fixed}. \quad (4.18)
$$

Given one of the $n = (n_1, n_2, n_3, n_4, n_5)$ for the fourteen decoupling limits listed in section 2.3, we set the chemical potentials to be given by (2.4), and then take the limit (4.18) of the letter partition function (4.17). The resulting letter partition functions for the twelve non-trivial decoupled theories are listed at the end of section 4.2. Given one of the decoupled letter partition functions $z(\tilde{x})$ we can then find the partition function for free SU(N) $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the decoupling limit as

$$
Z_{\tilde{\lambda}=0,N}(\tilde{x}) = \int [dU] \exp\left[\sum_{k=1}^{\infty} \frac{1}{k} z(\eta^{k+1}\tilde{x}^k) \left(\text{Tr}(U^k)\, \text{Tr}(U^{-k}) - 1\right)\right]. \quad (4.19)
$$

In the planar limit $N = \infty$ this reduces to

$$
\log Z_{\tilde{\lambda}=0,N=\infty}(\tilde{x}) = -\sum_{k=1}^{\infty} \log\left[1 - z(\eta^{k+1}\tilde{x}^k)\right]. \quad (4.20)
$$

We see from (4.20) that we have a Hagedorn singularity for $z(\tilde{x}) = 1$. This defines the Hagedorn temperature $\tilde{T}_{\mathrm{H}}^{(0)}$ for each of the twelve non-trivial decoupled theories. In the

end of section 4.2 we have listed $\tilde{T}_{\mathrm{H}}^{(0)}$ for each of the theories.[6] For the two $\mathrm{U}(1)$ theories $\tilde{T}$ can be arbitrarily large and the Hagedorn singularity is never reached.

4.2 Hagedorn temperature for small $\tilde{\lambda}$

In this section we consider small $\tilde{\lambda}$ and work out the Hagedorn temperature up to one-loop order for each of the decoupled theories. The results are presented as a list at the end of this section.

The general formula for the one-loop correction to Hagedorn temperature is given by [22, 6]

$$\delta\tilde{T}_{\mathrm{H}} = \tilde{\lambda} \left. \frac{\langle D_2(\tilde{x})\rangle}{\tilde{T}\frac{\partial z(\tilde{x})}{\partial \tilde{T}}} \right|_{\tilde{T}=\tilde{T}_{\mathrm{H}}^{(0)}} \tag{4.21}$$

where $\tilde{T}_{\mathrm{H}}^{(0)}$ is the free Hagedorn temperature of a specific theory, $z(\tilde{x})$ is the corresponding letter partition function and

$$\langle D_2(\tilde{x})\rangle = \sum_{A_1, A_2 \in \mathcal{A}} \tilde{x}^{d(A_1)+d(A_2)} \langle A_1 A_2 | D_2 | A_1 A_2 \rangle \tag{4.22}$$

is the expectation value of the corresponding one-loop dilatation operator [22, 6].

To compute $\langle D_2(\tilde{x})\rangle$ in the presence of chemical potentials for the R-charges and for the Cartan generators of the $\mathrm{SO}(4)$ symmetry we generalize the procedure used in [6]. In general, $\langle D_2(x, \omega_i, \Omega_i)\rangle$ corresponds to the expectation value of the one-loop dilatation operator D_2 acting on the product of two copies of the singleton representation $\mathcal{A} \times \mathcal{A}$. From eq. (2.20) we have that

$$\langle D_2(x, \omega_i, \Omega_i)\rangle = \sum_{j=0}^{\infty} h(j) \frac{V_j(x, \omega_i, \Omega_i)}{\left(1 + x^2 - 2x\cosh(\beta\omega_1)\right)\left(1 + x^2 - 2x\cosh(\beta\omega_2)\right)} \tag{4.23}$$

where $V_j(x, \omega_i, \Omega_i)$ can be computed using the results presented in [6] where in this case we define

$$F_{[k,p,q]}^{(j_L, j_R)} = W_{[k,p,q]} \sum_{m=-j_L}^{j_L} \sum_{\bar{m}=-j_R}^{j_R} \rho^m \bar{\rho}^{\bar{m}}, \qquad W_{[k,p,q]} \equiv \mathrm{Tr}_{[k,p,q]}\left(y_i^{J_i}\right), \tag{4.24}$$

with ρ and $\bar{\rho}$ defined in (4.6) and where the expressions of $W_{[k,p,q]}$ for the various representations are given in [6].

The general procedure described above allows us to compute $\langle D_2(x, \omega_i, \Omega_i)\rangle$ for $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with all five chemical potentials turned on. By taking the various decoupling limits we obtain expressions for $\langle D_2(\tilde{x})\rangle$ in each decoupled theory.[7]

---

[6]It is interesting to notice that some theories have the same free Hagedorn temperature and that the chemical potentials in these theories are all related by a permutation. The theories $\mathrm{SU}(1|2)$, $\mathrm{SU}(1,1|1)$ and $\mathrm{SU}(1,2|1)$ have for example all the same $\tilde{T}_{\mathrm{H}}^{(0)}$ and their critical chemical potentials $(n_1, n_2, n_3, n_4, n_5)$ are all given with some permutation of $(1, 1, \frac{1}{2}, \frac{1}{2}, 0)$.

[7]One can also compute $\langle D_2 \rangle$ in the decoupled theories using the general procedure found in [23].

We now have all the ingredients needed to find the one-loop correction to the Hagedorn temperature from eq. (4.21). We end this section with a list of results for the letter partition function, the expectation value of D_2 and the Hagedorn temperature up to one-loop order for all the non-trivial decoupled theories. The trivial theories are the bosonic $\mathrm{U}(1)$ with $z(\tilde{x}) = \tilde{x}$ and the fermionic $\mathrm{U}(1)$ with $z(\tilde{x}) = \tilde{x}^{\frac{3}{2}}$. In both of these theories D_2 vanishes and there is no Hagedorn singularity.

The $\mathrm{SU}(2)$ theory

$$z(\tilde{x}) = 2\tilde{x}, \quad \langle D_2(\tilde{x}) \rangle = \tilde{x}^2, \quad \tilde{T}_{\mathrm{H}} = \frac{1}{\log 2} + \frac{\tilde{\lambda}}{4\log 2} + \mathcal{O}(\tilde{\lambda}^2) \qquad (4.25)$$

The $\mathrm{SU}(1|1)$ theory

$$z(\tilde{x}) = \tilde{x} + \tilde{x}^{\frac{3}{2}}, \quad \langle D_2(\tilde{x}) \rangle = \tilde{x}^{\frac{5}{2}} + \tilde{x}^3, \quad \tilde{T}_{\mathrm{H}} = \tilde{T}_{\mathrm{H}}^{(0)} + \frac{2\tilde{T}_{\mathrm{H}}^{(0)}\tilde{\lambda}}{3 + 2e^{1/2\tilde{T}_{\mathrm{H}}^{(0)}}} + \mathcal{O}(\tilde{\lambda}^2), \quad (4.26)$$

$$\tilde{T}_{\mathrm{H}}^{(0)} = 1/\log\left[\frac{1}{3} - \frac{5}{3}\left(\frac{2}{11 + 3\sqrt{69}}\right)^{1/3} + \frac{1}{6}\left(11 + 3\sqrt{69}\right)^{1/3}\right] \qquad (4.27)$$

The $\mathrm{SU}(1|2)$ theory

$$z(\tilde{x}) = 2\tilde{x} + \tilde{x}^{\frac{3}{2}}, \quad \langle D_2(\tilde{x}) \rangle = \left(\tilde{x} + \tilde{x}^{\frac{3}{2}}\right)^2, \quad \tilde{T}_{\mathrm{H}} = \frac{1}{\log\frac{2}{3-\sqrt{5}}}\left(1 + \frac{4}{5 + 3\sqrt{5}}\tilde{\lambda} + \mathcal{O}(\tilde{\lambda}^2)\right)$$

$$(4.28)$$

The $\mathrm{SU}(2|3)$ theory

$$z(\tilde{x}) = 3\tilde{x} + 2\tilde{x}^{\frac{3}{2}}, \quad \langle D_2(\tilde{x}) \rangle = 3\tilde{x}^2 + 6\tilde{x}^{\frac{5}{2}} + 3\tilde{x}^3, \quad \tilde{T}_{\mathrm{H}} = \frac{1}{\log 4}\left(1 + \frac{3}{8}\tilde{\lambda} + \mathcal{O}(\tilde{\lambda}^2)\right)$$

$$(4.29)$$

The bosonic $\mathrm{SU}(1,1)$ theory

$$z(\tilde{x}) = \frac{\tilde{x}}{1 - \tilde{x}}, \quad \langle D_2(\tilde{x}) \rangle = -\frac{\tilde{x}^2 \log(1 - \tilde{x})}{(1 - \tilde{x}^2)^2}, \quad \tilde{T}_{\mathrm{H}} = \frac{1}{\log 2} + \frac{1}{2}\tilde{\lambda} + \mathcal{O}(\tilde{\lambda}^2) \qquad (4.30)$$

The fermionic $\mathrm{SU}(1,1)$ theory

$$z(\tilde{x}) = \frac{\tilde{x}^{\frac{3}{2}}}{1 - \tilde{x}}, \quad \langle D_2(\tilde{x}) \rangle = -\frac{\tilde{x}^2 \log(1 - \tilde{x})}{(1 - \tilde{x}^2)^2}, \quad \tilde{T}_{\mathrm{H}} = \tilde{T}_{\mathrm{H}}^{(0)} + \frac{3e^{1/2\tilde{T}_{\mathrm{H}}^{(0)}}\tilde{\lambda}}{3e^{1/\tilde{T}_{\mathrm{H}}^{(0)}} - 1} + \mathcal{O}(\tilde{\lambda}^2),$$

$$(4.31)$$

with $\tilde{T}_{\mathrm{H}}^{(0)}$ the same as in eq. (4.27).

The $\mathrm{SU}(1,1|1)$ theory

$$z(\tilde{x}) = \frac{\tilde{x}}{1 - \sqrt{\tilde{x}}}, \quad \langle D_2(\tilde{x}) \rangle = -\frac{\tilde{x}^{\frac{3}{2}} \log(1 - \tilde{x})}{(1 - \sqrt{\tilde{x}})^2}, \quad \tilde{T}_{\mathrm{H}} = \frac{1}{\log\frac{2}{3-\sqrt{5}}} + \frac{1}{\sqrt{5}}\tilde{\lambda} + \mathcal{O}(\tilde{\lambda}^2)$$

$$(4.32)$$

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The $\mathrm{SU}(1,1|2)$ **theory**

$$z(\tilde{x}\ ) = \frac{2\tilde{x}}{1 - \sqrt{\tilde{x}}}, \quad \langle D_2(\tilde{x})\rangle = -\frac{\tilde{x}(1 + \sqrt{\tilde{x}})^2 \log(1 - \tilde{x})}{(1 - \sqrt{\tilde{x}})^2},$$

$$\tilde{T}_{\mathrm{H}} = \frac{1}{2\log 2} + \frac{3\log\frac{4}{3}}{4\log 2}\tilde{\lambda} + \mathcal{O}(\tilde{\lambda}^2) \tag{4.33}$$

The $\mathrm{SU}(1,2)$ **theory**

$$z(\tilde{x}) = \frac{\tilde{x}^2}{(1 - \tilde{x})^2}, \quad \langle D_2(\tilde{x})\rangle = \frac{\tilde{x}^3 + (\tilde{x}^2 - 2\tilde{x}^3)\log(1 - \tilde{x})}{(1 - \tilde{x})^4},$$

$$\tilde{T}_{\mathrm{H}} = \frac{1}{\log 2} + \frac{\tilde{\lambda}}{2\log 2} + \mathcal{O}(\tilde{\lambda}^2) \tag{4.34}$$

The $\mathrm{SU}(1,2|1)$ **theory**

$$z(\tilde{x}) = \frac{\tilde{x}^{\frac{3}{2}}}{(1 - \sqrt{\tilde{x}})^2(1 + \sqrt{\tilde{x}})}, \quad \langle D_2(\tilde{x})\rangle = \frac{\tilde{x}^{\frac{5}{2}} - (\tilde{x}^{\frac{5}{2}} + \tilde{x}^2 - \tilde{x}^{\frac{3}{2}})\log(1 - \tilde{x})}{(1 - \sqrt{\tilde{x}})^2(1 - \tilde{x})^2},$$

$$\tilde{T}_{\mathrm{H}} = \frac{1}{\log\frac{2}{3 - \sqrt{5}}}\left(1 + \sqrt{\frac{2}{5}}(3 - \sqrt{5})\,\tilde{\lambda} + \mathcal{O}(\tilde{\lambda}^2)\right) \tag{4.35}$$

The $\mathrm{SU}(1,2|2)$ **theory**

$$z(\tilde{x}) = \frac{\tilde{x}}{(1 - \sqrt{\tilde{x}})^2}, \quad \langle D_2(\tilde{x})\rangle = \frac{\tilde{x}^2 + (\tilde{x} - 2\tilde{x}^{\frac{3}{2}})\log(1 - \tilde{x})}{(1 - \sqrt{\tilde{x}})^4},$$

$$\tilde{T}_{\mathrm{H}} = \frac{1}{2\log 2} + \frac{\tilde{\lambda}}{4\log 2} + \mathcal{O}(\tilde{\lambda}^2) \tag{4.36}$$

The $\mathrm{SU}(1,2|3)$ **theory**

$$z(\tilde{x}) = \frac{3\tilde{x} - \tilde{x}^{\frac{3}{2}}}{(1 - \sqrt{\tilde{x}})^2}, \tag{4.37}$$

$$\langle D_2(\tilde{x})\rangle = \frac{\tilde{x}^{\frac{3}{2}}}{(1 - \tilde{x})^4}\left[\left(1 + 6\tilde{x}^{\frac{1}{2}} + 15\tilde{x} + 20\tilde{x}^{\frac{3}{2}} + 21\tilde{x}^2 + 6\tilde{x}^{\frac{5}{2}} - 19\tilde{x}^3 + 10\tilde{x}^4\right)\right.$$

$$\left. + \left(1 + 3\tilde{x}^{\frac{1}{2}} - 2\tilde{x} - 19\tilde{x}^{\frac{3}{2}} - 24\tilde{x}^2 - 19\tilde{x}^{\frac{5}{2}} - 4\tilde{x}^3 + 3\tilde{x}^{\frac{7}{2}} + \tilde{x}^4\right)\log(1 - \tilde{x})\right],$$

$$\tilde{T}_{\mathrm{H}} = \frac{1}{\log\frac{2}{7 - 3\sqrt{5}}}\left(1 - \frac{16[201341 - 90043\sqrt{5} + (262\sqrt{5} - 586)\log\frac{3\sqrt{5} - 5}{2}]}{5(5 - 3\sqrt{5})^4}\tilde{\lambda} + \mathcal{O}(\tilde{\lambda}^2)\right)$$

4.3 Large $\tilde{\lambda}$ limit of theories containing scalars

In this section we use the low energy spectrum (3.10)–(3.11) obtained from the general Bethe ansatz in the thermodynamic limit to study the large $\tilde{\lambda}$ Hagedorn temperature for the nine decoupled theories that contain scalars. The three remaining theories are discussed in section 4.5. This limit of the Hagedorn temperature has been calculated for

the decoupled SU(2) theory in [7] and for the SU(2) theory coupled to a magnetic field in [8]. We will use the same methods here to obtain a general expression for the large $\tilde{\lambda}$ Hagedorn temperature that is valid for all the nine non-trivial theories that contain at least one scalar. The Hagedorn temperature will depend on the numbers given in table 8.

There is a direct connection between the Hagedorn temperature in the decoupled theories and the free energy per site of the corresponding spin chain model in the thermodynamic limit [7]. For all the decoupled theories that contain a scalar we consider the function

$$V(\tilde{\beta}) \equiv \lim_{L \to \infty} \frac{1}{L} \log \left[\text{Tr}_L \left(e^{-\tilde{\beta}(H-L)} \right) \right]. \tag{4.38}$$

The limit is finite since $V(\tilde{\beta})$ is related to the thermodynamic limit of the free energy per site f by $V(\tilde{\beta}) = -\tilde{\beta} f(\tilde{\beta})$. The spectrum of the Hamiltonian is given in eq. (3.10). Note that in the definition of $V(\tilde{\beta})$ we subtract from the Hamiltonian the constant contribution L coming from D_0 but include the other contributions from D_0 that depend on the state of the spin chain. If we view $\tilde{\lambda} D_2$ as the Hamiltonian of the spin chain, then these additional terms from D_0 can in most cases be viewed as a coupling to an external magnetic field as in [8].

For all of the decoupled theories the partition function is given by

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty} \sum_{L=1}^{\infty} \frac{1}{n} e^{-n\tilde{\beta}L} \text{Tr}_L \left(e^{-n\tilde{\beta}(H-L)} \right) \tag{4.39}$$

for any value of $\tilde{\lambda}$. For large L we have that

$$e^{-n\tilde{\beta}L} \text{Tr}_L \left(e^{-n\tilde{\beta}(H-L)} \right) \simeq \exp \left(-n\tilde{\beta}L + LV(n\tilde{\beta}) \right). \tag{4.40}$$

From this observation we see that the Hagedorn temperature $\tilde{T}_{\text{H}} = 1/\tilde{\beta}_{\text{H}}$ for any of value of $\tilde{\lambda}$ is determined by the equation [7]

$$\tilde{\beta}_{\text{H}} = V(\tilde{\beta}_{\text{H}}). \tag{4.41}$$

We use this general equation for the Hagedorn temperature below to find the Hagedorn temperature for large $\tilde{\lambda}$. In section 4.4 we give an exact expression for $V(\tilde{\beta})$ for the SU(1|1) theory and use that to obtain the Hagedorn temperature for small $\tilde{\lambda}$ in that case as well.

In the following we use our knowledge of the low energy spectrum (3.10) to obtain the Hagedorn temperature for large $\tilde{\lambda}$. Recall that the low energy spectrum can be written in the form of eq. (3.10) using the number operators $M_n, N_n,$ and F_n. In order to find $V(\tilde{\beta})$ we are interested in the large L behavior of

$$\text{Tr}_L \left(e^{-\tilde{\beta}(H-L)} \right) = \sum_{\{M_n\}} \sum_{\{N_n\}} \sum_{\{F_n\}} \int_{-\frac{1}{2}}^{\frac{1}{2}} du \exp \left\{ -\tilde{\beta}(H-L) + 2\pi i u P \right\} \tag{4.42}$$

where $H - L$ is given by eq. (3.10) and the integration over u has been introduced to impose the zero momentum constraint in the spectrum (3.11). Evaluating the sums over

the number operators with M_n and N_n ranging from zero to infinity and F_n from zero to one, we get

$$\text{Tr}_L \left(e^{-\tilde{\beta}(H-L)} \right) = \tag{4.43}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} du \prod_{n \in \mathbb{Z}} \frac{\left(1 + \exp\left(\frac{2\pi^2 \tilde{\beta}\tilde{\lambda}}{L^2} n^2 - \frac{\tilde{\beta}}{2} + 2\pi i un \right) \right)^c}{\left(1 - \exp\left(\frac{2\pi^2 \tilde{\beta}\tilde{\lambda}}{L^2} n^2 + 2\pi i un \right) \right)^a \left(1 - \exp\left(\frac{2\pi^2 \tilde{\beta}\tilde{\lambda}}{L^2} n^2 - \tilde{\beta} + 2\pi i un \right) \right)^b}.$$

Analysis similar to the one in [7] shows that the leading contribution for $L \gg 1$ comes from $u = 0$ and that it is given by

$$\text{Tr}_L \left(e^{-\tilde{\beta}(H-L)} \right) \sim \exp\left(\frac{L}{\sqrt{2\pi\tilde{\lambda}\tilde{\beta}}} \sum_{p=1}^{\infty} \frac{a + b\left(e^{-\tilde{\beta}} \right)^p - c\left(-e^{\tilde{\beta}/2} \right)^p}{p^{3/2}} \right). \tag{4.44}$$

Using this in eq. (4.38) we arrive at

$$V(\tilde{\beta}) = \frac{1}{\sqrt{2\pi\tilde{\beta}\tilde{\lambda}}} \left[a\,\zeta(3/2) + b\,\text{Li}_{3/2}\left(e^{-\tilde{\beta}} \right) - c\,\text{Li}_{3/2}\left(-e^{-\tilde{\beta}/2} \right) \right] \tag{4.45}$$

where $\zeta(x)$ is the Riemann zeta function and $\text{Li}_n(x)$ is the Polylogarithm function. We can now solve the equation $V(\tilde{\beta}_{\text{H}}) = \tilde{\beta}_{\text{H}}$ to get the Hagedorn temperature for large $\tilde{\lambda}$

$$\tilde{T}_{\text{H}} = \left(\frac{2\sqrt{2\pi}}{(2a + 2b + (2 - \sqrt{2})c)\,\zeta(3/2)} \right)^{2/3} \tilde{\lambda}^{1/3} \tag{4.46}$$

This expression is valid for all theories that contain at least one scalar. The numbers a, b, c for each such theory are given in table 8. Note that (4.46) correctly reduces to the result obtained for the SU(2) decoupled theory in [7] for $a = 1$ and $b = c = 0$.

4.4 The SU(1|1) theory as a magnetic XX Heisenberg spin chain

In this section we rewrite the Hamiltonian of the decoupled SU(1|1) theory as a Heisenberg $XX_{1/2}$ spin chain coupled to an external magnetic field. The spin chain model is exactly solvable and using known results on the free energy we can in principle obtain the Hagedorn temperature for any value of $\tilde{\lambda}$. We demonstrate how the Hagedorn temperature can be obtained to arbitrary order in small $\tilde{\lambda}$ and for large $\tilde{\lambda}$ we verify that the exact result agrees with our Bethe ansatz method to obtain $V(\tilde{\beta})$.

Following [26], we rewrite the D_2 part of the SU(1|1) Hamiltonian in spin chain form by expressing it in terms of the three Pauli matrices $\sigma^1, \sigma^2, \sigma^3$ as

$$D_2 = \frac{1}{2} \sum_{j=1}^{L} (1 - \Pi_{j,j+1}) = \sum_{j=1}^{L} \frac{1}{2} \left((1 - \sigma_j^3) - \frac{1}{2}(\sigma_j^1 \sigma_{j+1}^1 + \sigma_j^2 \sigma_{j+1}^2) \right). \tag{4.47}$$

In spin chain language the bosonic partons Z are spin-up spinors and the fermionic partons χ_1 are spin-down. The D_0 part of the Hamiltonian can similarly be expressed in terms of
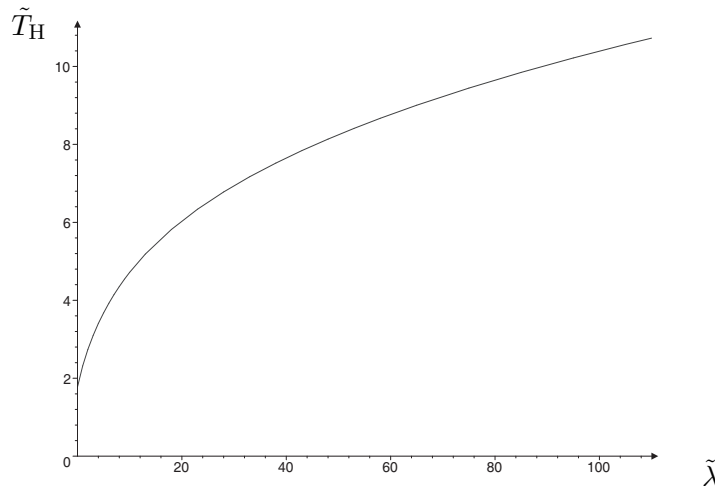
Figure 1: Hagedorn temperature $\tilde{T}_{\mathrm{H}}$ as a function of $\tilde{\lambda}$ for the SU(1|1) decoupled theory.

the Pauli matrices as

$$D_0 = L + \sum_{j=1}^{L} \frac{1}{4}(1 - \sigma_j^3) \tag{4.48}$$

and the full decoupled SU(1|1) Hamiltonian can therefore be written as

$$H_{\mathrm{SU(1|1)}} = L + \sum_{j=1}^{L} \left(\frac{1}{4}(1 + 2\tilde{\lambda}) \left(1 - \sigma_j^3\right) - \frac{\tilde{\lambda}}{4} \left(\sigma_j^1 \sigma_{j+1}^1 + \sigma_j^2 \sigma_{j+1}^2 \right) \right) \tag{4.49}$$

which is the Heisenberg $XX_{1/2}$ spin chain Hamiltonian with nearest neighbor coupling $\tilde{\lambda}/2$ in an external magnetic field of strength $(1+2\tilde{\lambda})/4$. This spin chain is exactly solvable and an expression for the free energy per site which is valid for all values of $\tilde{\lambda}$ is known [27]. In our notation this translates into

$$V(\tilde{\beta}) = -\frac{\tilde{\beta}}{4}(1 + 2\tilde{\lambda}) + \frac{1}{\pi} \int_0^{\pi} d\omega \log \left[2 \cosh \left(\frac{\tilde{\beta}}{2} \left\{ \frac{1}{2} + \tilde{\lambda}(1 - \cos \omega) \right\} \right) \right]. \tag{4.50}$$

From this function we can obtain the Hagedorn temperature for all values of $\tilde{\lambda}$ by employing the general equation (4.41) for the Hagedorn temperature. We have used this to plot the Hagedorn temperature $\tilde{T}_{\mathrm{H}}$ as a function of $\tilde{\lambda}$ in figure 1.

Hagedorn temperature for small $\tilde{\lambda}$. Let us first verify that can we match the $\tilde{\lambda} \to 0$ limit of the above considerations with the free Hagedorn temperature computation in section 4.1. We immediately get the condition

$$\tilde{\beta}_{\mathrm{H}} = V(\tilde{\beta}_{\mathrm{H}}) \Big|_{\tilde{\lambda}=0} = -\frac{\tilde{\beta}_{\mathrm{H}}}{4} + \log \left(2 \cosh \frac{\tilde{\beta}_{\mathrm{H}}}{4} \right) \tag{4.51}$$

which is equivalent to

$$e^{-\tilde{\beta}_{\rm H}} + e^{-3\tilde{\beta}_{\rm H}/2} = 1. \tag{4.52}$$

This is the same equation as obtained from the free letter partition function of the SU(1|1) theory in Secton 4.2 and the free Hagedorn temperature $\tilde{T}_{\rm H}^{(0)}$ is given in eq. (4.27). Equipped with the exact expression for $V(\tilde{\beta})$ we can now go further and obtain higher loop corrections for the Hagedorn temperature. Expanding eq. (4.50) for small $\tilde{\lambda}$ yields

$$V(\tilde{\beta}) = -\frac{\tilde{\beta}}{4} + \log 2 \cosh\left(\tilde{\beta}/4\right) + \frac{\tilde{\lambda}\tilde{\beta}}{2}\left(\tanh(\tilde{\beta}/4) - 1\right) + \frac{3\tilde{\beta}^2\tilde{\lambda}^2}{16}{\rm sech}^2\left(\tilde{\beta}/4\right) + O\left(\tilde{\lambda}^3\right) \tag{4.53}$$

and solving the equation $\tilde{\beta}_{\rm H} = V(\tilde{\beta}_{\rm H})$ then gives the Hagedorn temperature to this order

$$\tilde{T}_{\rm H} = \tilde{T}_{\rm H}^{(0)} + \frac{2\tilde{T}_{\rm H}^{(0)}\tilde{\lambda}}{3 + 2e^{1/2\tilde{T}_{\rm H}^{(0)}}} - \frac{e^{b/2}\left(17 + 28e^{b/2} + 12e^b\right)\tilde{\lambda}^2}{2\left(1 + e^{b/2}\right)\left(3 + 2e^{b/2}\right)^3} + O\left(\tilde{\lambda}^3\right) \tag{4.54}$$

where we have introduced the short hand notation $b = 1/\tilde{T}_{\rm H}^{(0)}$ to simplify the two-loop term. It is a comforting check that the one-loop term is precisely the same as found in eq. (4.26). Using the spin chain method we can easily obtain $\tilde{T}_{\rm H}$ to arbitrarily high order in $\tilde{\lambda}$.

Hagedorn temperature for large $\tilde{\lambda}$. From eq. (4.45) we already know the leading behavior of $V(\tilde{\beta})$ for large $\tilde{\lambda}$ and large L. As a check of that result we can extract the large $\tilde{\lambda}$ behavior of the exact function $V(\tilde{\beta})$ in eq. (4.50) and compare the two. From eq. (4.46) we know that $\tilde{\beta}_{\rm H} \sim \tilde{\lambda}^{-1/3}$ for large $\tilde{\lambda}$ and we are therefore interested in large $\tilde{\lambda}\tilde{\beta}$. In this limit we find

$$V(\tilde{\beta}) \simeq \frac{1}{\pi}\int_0^\pi d\omega \log\left[1 + \exp\left(-\tilde{\lambda}\tilde{\beta}(1 - \cos\omega)\right)\right] \simeq \frac{(\sqrt{2} - 1)\zeta(3/2)}{\sqrt{4\pi\tilde{\lambda}\tilde{\beta}}}. \tag{4.55}$$

The leading contribution comes from integrating over small ω and we therefore used the saddle-point approximation to get the final result. This is the same expression as eq. (4.45) with $a = b = 0$, $c = 1$, and the polylogarithm expanded for $\tilde{\beta} \ll 1$.

4.5 Large $\tilde{\lambda}$ limit of the SU(1, 2) theory

In section 4.3 we found the large $\tilde{\lambda}$ behavior of the Hagedorn temperature $\tilde{T}_{\rm H}$ in the decoupled theories containing scalars. This was done by considering the low energy behavior of the spin chain with Hamiltonian $H - L$, where $H = D_0 + \tilde{\lambda}D_2$. In the following we shall see that in theories without scalars the low energy behavior of the spin chain Hamiltonian cannot be connected with the large $\tilde{\lambda}$ behavior. We illustrate this by considering the SU(1, 2) theory, which is particularly interesting since the decoupled states are states of pure Yang-Mills theory, as we explore further in section 6. We comment below on the consequence of our observations for obtaining a string dual of the SU(1, 2) theory.

For the SU(1, 2) we found in section 3.2 the spectrum (3.13). This spectrum is accurate in the limit when L is large and $\tilde{\lambda}$ is large, and for states with $H - (2 + 3\tilde{\lambda}/2)L$ not large. In analogy with (4.38) we define the function $V_{\mathrm{SU}(1,2)}(\tilde{\beta})$ as

$$V_{\mathrm{SU}(1,2)}(\tilde{\beta}) \equiv \lim_{L \to \infty} \frac{1}{L} \log \left[\mathrm{Tr}_L \left(e^{-\tilde{\beta}(H - (2 + \frac{3}{2}\tilde{\lambda})L)} \right) \right]. \tag{4.56}$$

Consider then the large $\tilde{\lambda}$ limit of (4.56). Using (3.13) we see that the spectrum of $H - (2 + \frac{3}{2}\tilde{\lambda})L$ consists of a magnetic part and a part which is proportional to $\tilde{\lambda}/L^2$. While the magnetic part does not receive finite size correction, the other part does. From this, one sees that the finite size corrections are suppressed in $V_{\mathrm{SU}(1,2)}(\tilde{\beta})$ provided that $\tilde{\lambda}\tilde{\beta} \gg 1$. Therefore, we can use the spectrum (3.13) to find that

$$V_{\mathrm{SU}(1,2)}(\tilde{\beta}) \simeq \frac{\sqrt{6}}{\sqrt{\pi \tilde{\lambda}\tilde{\beta}}} \mathrm{Li}_{3/2}\left(e^{-\tilde{\beta}}\right) \quad \text{for} \quad \tilde{\lambda}\tilde{\beta} \gg 1. \tag{4.57}$$

Now, consider the Hagedorn temperature for the SU(1, 2) theory in general. Following the argument of section 4.3 we see that the Hagedorn temperature $\tilde{T}_{\mathrm{H}} = 1/\tilde{\beta}_{\mathrm{H}}$ for any value of $\tilde{\lambda}$ is given by the equation

$$\left(2 + \frac{3}{2}\tilde{\lambda}\right) \tilde{\beta}_{\mathrm{H}} = V_{\mathrm{SU}(1,2)}(\tilde{\beta}_{\mathrm{H}}). \tag{4.58}$$

Take then the large $\tilde{\lambda}$ limit. We see first that eq. (4.58) becomes $3\tilde{\lambda}\tilde{\beta}_{\mathrm{H}} \simeq 2V_{\mathrm{SU}(1,2)}(\tilde{\beta}_{\mathrm{H}})$. Then, if we try to insert the approximation (4.57) for $V_{\mathrm{SU}(1,2)}(\tilde{\beta})$ we get the equation

$$\frac{3\sqrt{\pi}}{2\sqrt{6}}\left(\tilde{\lambda}\tilde{\beta}_{\mathrm{H}}\right)^{\frac{3}{2}} \simeq \mathrm{Li}_{3/2}\left(e^{-\tilde{\beta}_{\mathrm{H}}}\right). \tag{4.59}$$

However, $\mathrm{Li}_{3/2}(e^{-x})$ is a decreasing function of x bounded from above by $\mathrm{Li}_{3/2}(1) = \zeta(3/2)$. Thus, the relation (4.59) requires that $\tilde{\lambda}\tilde{\beta}_{\mathrm{H}}$ is of order one or smaller. Clearly, this conflicts with the approximation used to derive (4.57). Therefore, we cannot infer the behavior of the Hagedorn temperature $\tilde{T}_{\mathrm{H}}$ for large $\tilde{\lambda}$ using the result (4.57).

We can therefore conclude that the free magnon spectrum (3.13) does not correspond to the behavior of the SU(1, 2) theory for large $\tilde{\lambda}$. To understand the large $\tilde{\lambda}$ behavior of the SU(1, 2) one must therefore solve the full Bethe equations (3.4)–(3.5) for that theory. Therefore, contrary to the theories with scalars, the large $\tilde{\lambda}$ limit does not correspond to a free magnon limit of the spin chain for this decoupled theory.

That we cannot use the free spectrum (3.13) to approximate large $\tilde{\lambda}$ for the SU(1, 2) theory means that it is considerably harder to understand the SU(1, 2) decoupling limit on the string theory side in the AdS/CFT correspondence. In [7] it was found for the SU(2) theory how to obtain the spectrum and the Hagedorn temperature for large $\tilde{\lambda}$ from the string side. However, it is not clear how to find a similar match of the spectrum and Hagedorn temperature for the SU(1, 2) theory since it is not well understood how to obtain the full set of finite-size effects on the string side.

Note finally that the above considerations for the SU(1, 2) theory can be repeated for the other two decoupled theories without scalars, i.e. the fermionic SU(1, 1) and the

SU(1, 2|1) theories, with analogous results. Thus, also for these two theories the large $\tilde{\lambda}$ behavior is not linked to the free magnon limit of a spin chain.

5. Microcanonical version of the decoupling limits

The decoupling limits described in section 2 are taken of the partition function of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the grand canonical ensemble. It is highly useful to understand how the decoupling limits are taken in the microcanonical ensemble. In particular, this is necessary in order to translate these decoupling limits to the string side of the AdS/CFT correspondence. In section 5.1 we consider how to implement the decoupling limits in the microcanonical ensemble and in section 5.2 we use the microcanonical decoupling limits to identify, for any of the decoupling limits containing scalars, a regime of weakly coupled planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in which it corresponds to tree-level string theory.

Another reason why it is important to obtain an understanding of our decoupling limits in the microcanonical ensemble is that one can think of the decoupling limits solely in terms of gauge invariant operators of $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$. This is in contrast to the grand canonical ensemble in which the correct interpretation is rather in terms of the partition function which sums over states of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. Thus, both the microcanonical decoupling limits that we present below in section 5.1 and the "stringy regime" that we present in section 5.2 apply also to gauge-invariant operators of $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$.

5.1 Microcanonical limit

Let $n = (n_1, n_2, n_3, n_4, n_5)$ be given such that it fulfils the requirements described in section 2.1. Let J be defined as in (2.6). Then the decoupling limit of SU(N) $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the microcanonical ensemble is given as

$$\lambda \to 0, \quad \tilde{H} \equiv \frac{D - J}{\lambda} \text{ fixed}, \quad J, N \text{ fixed} \tag{5.1}$$

where D is the dilation operator which is expanded as (2.7) for small λ. We see that the limit (5.1) indeed is in the microcanonical ensemble since $\tilde{H}$ and J are linear combinations of the Cartan generators of $psu(2, 2|4)$. Analyzing the limit (5.1) we see that since $D \simeq D_0 + \lambda D_2$ for small λ only states with $D_0 = J$ survive and we get that $\tilde{H} = D_2$ for D_2 acting on the surviving states.

We first observe that the set of states/operators that we have after the decoupling limit are the ones with $D_0 = J$ for a given J. Thus, whereas for the grand canonical limit (2.13) we had all states with $D_0 = J$ for any choice of J, for the microcanonical limit we only have the subset of states corresponding to a particular fixed value of J. Therefore, the microcanonical decoupling limit (5.1) is seen to give a subset of the decoupled states that we get in the grand canonical decoupling limit (2.13).

We furthermore observe that while for the grand canonical limit (2.13) we have $D_0 + \tilde{\lambda} D_2$ as the effective Hamiltonian, we have $\tilde{H} = D_2$ as the effective Hamiltonian for the microcanonical limit (5.1). This is in accordance with the fact that we pick a fixed J in the decoupled theory, since we clearly have that the $D_0 + \tilde{\lambda} D_2$ Hamiltonian is equivalent to

choosing D_2 as the Hamiltonian if we keep D_0 fixed. We can therefore conclude that the two decoupling limits (2.13) and (5.1) give us the same decoupled theory in two different ensembles, i.e. in the (2.13) limit we end up in the canonical ensemble while in the (5.1) limit we end up in the microcanonical ensemble.

Another important point is that it follows from the commutation relation $[D_2, D_0] = 0$ in eq. (2.21) that $\tilde{H}$ commutes with J. This means that there are no interactions between states with different values of J, i.e. the subsector of the decoupled theory that we choose by fixing J is closed with respect to $\tilde{H}$.

We can thus conclude that the microcanonical decoupling limit (5.1) for a given $n = (n_1, n_2, n_3, n_4, n_5)$ leads to the same decoupled theory as the grand canonical limit (2.13). It is moreover clear that the same analysis applies concerning which decoupling limits one has, thus the list of decoupling limits of section 2.3 applies equally well to the microcanonical decoupling limit (5.1).

In the planar limit $N = \infty$ of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ and in the decoupling limit (5.1), with $n = (n_1, n_2, n_3, n_4, n_5)$ chosen from the list in section 2.3, the spectrum of $\tilde{H} = D_2$ is given by (3.3) with the Bethe roots determined by (3.4)–(3.5). Thus, we have the full spectrum for the decoupled theory in the planar limit and each of the limits of section 2.3 correspond to an integrable spin chain.

However, there is a subtle issue in applying the spin chain picture to the microcanonical decoupling limit (5.1). In the microcanonical limit (5.1) we fix J, whereas when applying the Bethe equation (3.4) we consider a certain length L of the spin chain. However, in general the length L is not fixed for a given J. For instance, in the bosonic SU(1, 1) limit the operators $\mathrm{Tr}(ZZZ)$ and $\mathrm{Tr}(Zd_1Z)$ both have $J = 3$ while L is 3 and 2, respectively. Therefore, when applying the Bethe ansatz technique, one should divide the decoupled theory into the different subsectors according to the possible values of L, and then apply the Bethe ansatz technique separately for these subsectors. It is however necessary for this to work that there are no interactions between the subsectors of different lengths. That this is the case can be seen by the fact that the D_2 operator cannot change the length of a state. One way to see this is to observe that the length operator is $L = 1 - C$, where C is the central charge of the $u(2, 2|4)$ algebra, as reviewed in appendix A. From this fact it is easy to check that one has $[D_2, L] = 0$, hence D_2 does not change the length.

5.2 Regimes of $\mathcal{N} = 4$ SYM with stringy behavior

In [7] it was found that for the SU(2) decoupling limit (see section 2.3) one can match the spectrum and Hagedorn temperature as found from the gauge theory and string theory sides when $\tilde{\lambda} \to \infty$.

One of the reasons behind the successful match of [7] is that $\tilde{\lambda} = \lambda/(1 - \Omega)$ works as an effective 't Hooft coupling in the decoupled theory. This can for example be seen from the fact that the $\tilde{\lambda}^n$ contribution to the Hagedorn temperature $\tilde{T}_H$ origins from part of the n-loop diagram for $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. Therefore, taking the large $\tilde{\lambda}$ limit can be seen as taking the strong coupling limit. However, since the effective Hamiltonian is $D_0 + \tilde{\lambda} D_2$ this can be accomplished in a controllable manner. Thus, in this sense one can say that

the successful match between gauge theory and string theory in [7] is due to the fact that it was found how to take the strong coupling limit in a controllable way.

We expect that the match between gauge theory and string theory in the large $\tilde{\lambda}$ limit works for all the 9 theories in the list of limits in section 2.3 that include scalars, i.e. the $SU(2)$, $SU(1|1)$, $SU(1|2)$, $SU(2|3)$, bosonic $SU(1,1)$, $SU(1,1|1)$, $SU(1,1|2)$, $SU(1,2|2)$ and $SU(1,2|3)$ limits. This is in accordance with the results of section 3.2 where it is found that for large $\tilde{\lambda}$ the spectrum is string-like.

The question of this section is then in which regime of planar $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$ do we see stringy behavior, given any of these 9 decoupling limits with scalars, i.e. how do we translate the large $\tilde{\lambda}$ limit, which makes sense in the grand canonical ensemble, to a statement about $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$ in the microcanonical ensemble.

Now, taking the large $\tilde{\lambda}$ limit for the effective Hamiltonian $D_0 + \tilde{\lambda}D_2$ corresponds to considering the low energy states for the D_2 operator (for the decoupled theories with scalars). In other words, for large $\tilde{\lambda}$ we consider states with D_2 of order $1/\tilde{\lambda}$ so that $\tilde{\lambda}D_2$ is of order one. Since $(D-J)/\lambda$ approaches D_2 in the limit (5.1) we see that we should have $(D-J)/\lambda$ to be of order $1/\tilde{\lambda}$. Thus, we need that $|D-J| \ll \lambda$. The limit (5.1) also requires $\lambda \ll 1$ and $|D-J| \ll 1$, and in addition we need large J to see string-like states, so combining these ingredients we get that the large $\tilde{\lambda}$ limit corresponds to probing the regime

$$|D-J| \ll \lambda \ll 1, \quad J \gg 1. \tag{5.2}$$

Thus, for $n = (n_1, n_2, n_3, n_4, n_5)$ corresponding to one of the nine non-trivial decoupling limits with scalars listed in section 2.3, we have identified the regime (5.2) for which planar $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$ has a string-like spectrum, and for which we expect to be able to match gauge theory and string theory. In particular, we expect to find semi-classical string states in planar $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$ in the regime (5.2).

Note that it is clear from (5.2) that the $|D-J| \ll \lambda \ll 1$ requirement means that only states with $D_0 = J$ can be present. Thus, (5.2) is a alternative way of representing the perhaps most interesting part of our decoupled theories without resorting to limits.

As we comment on further in the Conclusions in section 7, it would be highly interesting to examine the regimes (5.2) of $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$ further. In these regimes one can hope to find precise matches between weakly coupled gauge theory and string theory.

Finally, it is important to explain why we only consider the nine theories with scalars, and not the fermionic $SU(1,1)$, the $SU(1,2)$ and the $SU(1,2|1)$ theories. This is due to the presence of the cL term in the dispersion relation (3.3), with non-zero c. This means that for large $\tilde{\lambda}$ there is a $\tilde{\lambda}cL$ term in $\tilde{\lambda}D_2$. With such a term one cannot connect having large $\tilde{\lambda}$ to the low energy behavior of D_2. Hence the regime (5.2) does not apply for these three theories. This is another manifestation of the fact that the free limit of the spin chains and the large $\tilde{\lambda}$ limit are not connected for these three theories, as already discussed for the large $\tilde{\lambda}$ limit of the $SU(1,2)$ theory in section 4.5.

6. A decoupling limit of pure Yang-Mills theory

In this section we consider a new decoupling limit of pure Yang-Mills theory (YM) on

$\mathbb{R} \times S^3$. In the planar limit, the pure YM theory reduces in the decoupling limit to a fully integrable spin chain. The limit is analogous to the SU(1, 2) limit of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ as found in section 2. We furthermore write down a microcanonical version of the limit which also applies to gauge-invariant operators of pure YM on $\mathbb{R}^4$.

The pure YM Lagrangian is invariant classically under conformal transformations. Thus, it has the conformal group in four dimensions SO(2, 4) $\simeq$ SU(2, 2) as symmetry group. However, contrary to $\mathcal{N} = 4$ SYM, pure YM is not a conformal theory since the conformal symmetry is broken by quantum corrections. Specifically, the beta function $\beta(\lambda)$ for the 't Hooft coupling of pure YM becomes non-zero at second order in the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N/(4\pi^2)$ [28, 29].

Nevertheless, since the beta function is non-zero only at 2-loop order, we can regard pure YM as being a conformal theory when considering only the tree-level and one-loop diagrams. And this will be enough to formulate a decoupling limit for pure YM, based on the same considerations as for $\mathcal{N} = 4$ SYM.

The Cartan generators of the conformal group SO(2, 4) are the dilatation operator $D^{(\mathrm{YM})}$ and the two Cartan generators S_1 and S_2 for the SO(4) subgroup. For small 't Hooft coupling we can expand the dilatation operator as $D^{(\mathrm{YM})} = D_0^{(\mathrm{YM})} + \lambda D_2^{(\mathrm{YM})} + \mathcal{O}(\lambda^2)$ where $D_0^{(\mathrm{YM})}$ is the bare scaling dimension and $D_2^{(\mathrm{YM})}$ gives the one-loop anomalous dimension (computed in [30]). We write the temperature as $T = 1/\beta$ and the chemical potentials corresponding to S_1 and S_2 as ω_1 and ω_2.

Since pure YM is conformally invariant to one-loop order we can employ the state/operator correspondence relating states of pure YM on $\mathbb{R} \times S^3$ to gauge-invariant operators of pure YM on $\mathbb{R}^4$. The set of gauge-invariant operators of pure YM consists of the linear combinations of multi-trace operators that can be constructed using the set of letters consisting of the 6 gauge field strength components and the descendants obtained by applying the covariant derivative. The gauge field strength and covariant derivative components transform as in $\mathcal{N} = 4$ SYM, thus one can use tables 1 and 6 also for pure YM, if one ignores the SU(4) part.

In the following we take the two chemical potentials to be equal $\omega_1 = \omega_2 = \omega$ and consider the decoupling limit of pure YM with gauge group SU(N) on $\mathbb{R} \times S^3$ given by

$$\beta \to \infty, \quad \tilde{\beta} \equiv \beta(1 - \omega) \text{ fixed}, \quad \tilde{\lambda} \equiv \frac{\lambda}{1 - \omega} \text{ fixed}, \quad N \text{ fixed}. \tag{6.1}$$

By the same arguments as in section 2.1, one sees that the complete partition function in the grand canonical ensemble of pure YM with gauge group SU(N) on $\mathbb{R} \times S^3$ in the limit (6.1) reduces to

$$Z_{\tilde{\lambda}, N}(\tilde{\beta}) = \mathrm{Tr}_{\mathcal{H}} \left[\exp \left\{ -\tilde{\beta} \left(D_0^{(\mathrm{YM})} + \tilde{\lambda} D_2^{(\mathrm{YM})} \right) \right\} \right] \tag{6.2}$$

where $\mathcal{H}$ is the set of gauge-invariant operators (or the corresponding states) obeying $D_0^{(\mathrm{YM})} = S_1 + S_2$. Thus, $\mathcal{H}$ consists of any linear combination of multi-trace operators that can be written using the letters $d_1^m d_2^k \bar{F}_+$. This set of letters transforms in the $[0, -3]$ representation of the $su(1, 2)$ algebra, hence the decoupled theory has a SU(1, 2) symmetry.

This can be seen by employing the same arguments as for the SU(1, 2) limit of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$.

We thus see that in the decoupling limit (6.1) only the operators in $\mathcal{H}$, made from the letters $d_1^m d_2^k \bar{F}_+$, contribute to the partition function. All the other gauge-invariant operators of pure YM are decoupled. Moreover, we have an effective Hamiltonian $D_0^{(\mathrm{YM})} + \tilde{\lambda} D_2^{(\mathrm{YM})}$. From [30] we have that $D_2^{(\mathrm{YM})} = D_2 - \frac{11}{12} L$ for operators in $\mathcal{H}$, where D_2 is the truncation of the one-loop contribution to the dilatation operator of $\mathcal{N} = 4$ SYM in the SU(1, 2) decoupled theory. Using this, we can translate the results for the SU(1, 2) theory in $\mathcal{N} = 4$ SYM to pure YM in the decoupling limit (6.1).

We now turn to the planar limit of pure YM on $\mathbb{R} \times S^3$. Here we can focus on single-trace operators, and they can be interpreted as states in a spin chain. It has been shown in [30] that planar pure YM is integrable to one loop when restricting to chiral operators. Since the set of operators $\mathcal{H}$ is chiral, we inherit the integrability for the full chiral sector in our decoupling limit (6.1). Moreover, since our full Hamiltonian $D_0^{(\mathrm{YM})} + \tilde{\lambda} D_2^{(\mathrm{YM})}$ only contains tree-level and one-loop terms, our decoupled theory is fully integrable.

Since the decoupled theory has a SU(1, 2) symmetry, the spectrum can be found from an SU(1, 2) spin chain. In detail, the spectrum follows from the dispersion relation

$$D_2^{(\mathrm{YM})} = \frac{1}{2} \sum_{k=1}^{K} \frac{|V_{j_k}|}{u_k^2 + \frac{1}{4} V_{j_k}^2} + \frac{7}{12} L \tag{6.3}$$

along with the Bethe equation (3.4), inserting here that we are in the $[0, -3]$ representation of $su(1, 2)$, and the cyclicity condition (3.5) with $U = 1$. This gives the full spectrum of pure YM on $\mathbb{R} \times S^3$ in the decoupling limit (6.1).

We can furthermore follow our computations of section 4 and obtain the thermodynamics of the decoupled theory. First, the letter partition function for pure YM is given by (4.10). Taking the limit (6.1) of this we get

$$z(\tilde{x}) = \frac{\tilde{x}^2}{(1 - \tilde{x})^2} \tag{6.4}$$

as for the SU(1, 2) limit of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. Computing furthermore the expectation value of $D_2^{(\mathrm{YM})}$, we get

$$\langle D_2^{(\mathrm{YM})}(\tilde{x}) \rangle = \frac{\tilde{x}^2 \left[\left(1 - \frac{11}{12}\tilde{x}\right) \tilde{x} + (1 - 2\tilde{x}) \log(1 - \tilde{x}) \right]}{(1 - \tilde{x})^4}. \tag{6.5}$$

Using (6.4) and (6.5) along with (4.21) we get the Hagedorn temperature in the decoupled theory to first order in $\tilde{\lambda}$

$$\tilde{T}_{\mathrm{H}} = \frac{1}{\log 2} + \tilde{\lambda} \frac{13}{48 \log 2} + \mathcal{O}(\tilde{\lambda}^2). \tag{6.6}$$

Turning instead to large $\tilde{\lambda}$, we run into the same difficulties as encountered in section 4.5 for the SU(1, 2) decoupling limit of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. Defining the function

$$V_{(\mathrm{YM})}(\tilde{\beta}) \equiv \lim_{L \to \infty} \frac{1}{L} \log \left[\mathrm{Tr}_L \left(e^{-\tilde{\beta} \tilde{\lambda} \left(D_2^{(\mathrm{YM})} - \frac{7}{12} L \right)} \right) \right] \tag{6.7}$$

we see that in general the Hagedorn temperature satisfies the equation

$$\left(2 + \frac{7}{12}\tilde{\lambda}\right)\tilde{\beta}_{\mathrm{H}} = V_{(\mathrm{YM})}(\tilde{\beta}_{\mathrm{H}}). \qquad (6.8)$$

However, while the left-hand side of (6.8) goes to infinity as $\tilde{\lambda}\tilde{\beta}_{\mathrm{H}} \to \infty$, the right-hand side goes to zero, in parallel with the analysis of section 4.5. Thus, we cannot use the free limit of the spin chain to infer the large $\tilde{\lambda}$ behavior of the Hagedorn temperature. Following the discussion in section 4.5, this has the consequence that a string dual of pure YM on $\mathbb{R} \times S^3$ in the decoupling limit (6.1) will be difficult to find, since one cannot consider a limit wherein the world-sheet theory of the strings is free. More generally, this suggests that a string dual of pure YM will be difficult to attain.

As for the decoupling limits of $\mathcal{N} = 4$ SYM, we can write the decoupling limit (6.1) as a decoupling limit in the microcanonical ensemble, following section 5. This microcanonical decoupling limit of pure YM with gauge group $\mathrm{SU}(N)$ takes the form

$$\lambda \to 0, \quad \tilde{H} \equiv \frac{D^{(\mathrm{YM})} - S_1 - S_2}{\lambda} \text{ fixed}, \quad S_1 + S_2, \, N \text{ fixed}. \qquad (6.9)$$

This limit can also be thought of as a decoupling limit for gauge-invariant operators of pure YM on $\mathbb{R}^4$.

The search for integrable structures in pure YM and QCD has received considerable attention recently [31, 30]. In [30] the full one-loop anomalous dimension matrix has been computed and studied, finding a large integrable structure in the chiral sectors. The decoupling limit (6.1) gives a decoupled sector which is a subsector of one of the chiral sectors. However, the advantage of our decoupling limit (6.1) is that after the limit we get a decoupled theory which is fully integrable. This enables us to study what happens in a strong coupling limit, which for the decoupled theory is $\tilde{\lambda} \to \infty$.

Finally, we remark that it was conjectured in [20] that the Hagedorn phase transition in weakly coupled pure YM on $\mathbb{R} \times S^3$ is continuously connected to the confinement/deconfinement transition in pure YM on $\mathbb{R}^4$. This suggest that our above results perhaps can be useful to learn more about the confinement/deconfiment transition in pure YM.

7. Discussion and conclusions

The general idea of this paper is to consider $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ near critical points with zero temperature and critical chemical potentials. Analyzing $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in such a near-critical region gives rise to fourteen different decoupled theories that are a good description of weakly coupled $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ near fourteen different critical points. The precise formulation of this is in terms of the decoupling limits (2.13) which are taken of the partition function in the grand canonical ensemble. Taking these limits we decouple physically fourteen different theories contained in $\mathcal{N} = 4$ SYM that are much simpler than the full theory but still share many of its interesting features. The chemical potentials that we have are the two chemical potentials for the SO(4) symmetry and the three chemical

potentials for the SU(4) R-symmetry. The analysis of the near-critical regions generalizes the one of [6] where only the R-symmetry chemical potentials were considered.

For each of the decoupled theories we found an effective Hamiltonian of the form $D_0 + \tilde{\lambda} D_2$. This Hamiltonian is valid for any value of $\tilde{\lambda}$, thus we can study the decoupled theory for any value of $\tilde{\lambda}$ since both D_0 and D_2 are known explicitly. We used this fact to study the planar limit, where for each of the fourteen theories D_2 is equivalent to a Hamiltonian for an integrable spin chain. In the theories with scalars we used this to determine the spectrum and the Hagedorn temperature in the limit of large $\tilde{\lambda}$. In this sense we see that we are able to take explicitly a strong coupling limit for these nine decoupled theories.

One of the decoupling limits gives rise to a decoupled theory with SU(1, 2|3) symmetry. We have shown that this particular theory contains all of the other thirteen decoupled theories. Note that this theory also contains the half-BPS operators of $\mathcal{N} = 4$ SYM since they all satisfy the relation $D = S_1 + S_2 + J_1 + J_2 + J_3$.

The SU(1|1) decoupled theory is particularly interesting since in the planar limit it corresponds to an exactly solvable spin chain, namely the Heisenberg $XX_{1/2}$ spin chain coupled to an external magnetic field. Thus, for this decoupled theory the exact partition function can be found. Using this we obtained an exact equation that determines the Hagedorn temperature as a function of $\tilde{\lambda}$, from which the small and large $\tilde{\lambda}$ expansions are easily infered.

Another interesting decoupled theory that we studied is the one with SU(1, 2) symmetry. For this theory, it is considerably harder to take the large $\tilde{\lambda}$ limit. This is seen by considering the planar limit, for which we find that in the free magnon spectrum of D_2 the ground state energy is moved up from zero to a value proportional to the length of the spin chain L, contrary to what happens for the nine non-trivial theories with scalars.

The pure YM decoupling limit (6.1) gives rise to a decoupled theory which is almost identical to the SU(1, 2) decoupled theory of $\mathcal{N} = 4$ SYM. This is interesting in view of the problems with taking the large $\tilde{\lambda}$ limit since they translate to the pure YM decoupled theory. This suggests that it is hard to find a string-dual of pure YM, since our results imply that one cannot find a regime in which the world-sheet theory is free.

We identified an equivalent formulation of the decoupling limits in terms of the microcanonical ensemble. This is important since it gives a better understanding of which regime of the theory we zoom in to when going near one of the critical points. We used in particular these insights to determine the regimes (5.2) of $\mathcal{N} = 4$ SYM in which we have string-like states.

Future directions and outlook. Inspired by the work [7, 8], one of the interesting future directions is to find the decoupling limits for type IIB strings on $\mathrm{AdS}_5 \times S^5$ that are dual to the gauge theory decoupling limits found in the present paper. We expect this to be possible for the nine decoupling limits for which the decoupled theories have scalars [32]. It would in particular be interesting to find Penrose limits consistent with the decoupling limits, enabling one to match the spectra on the gauge and string sides in the large $\tilde{\lambda}$ limit and for long operators.

Following [8] it would be interesting to examine the more general decoupling limits for which one obtains effective chemical potentials in the decoupled theories. For example for the $SU(1,2|3)$ limit one has four effective chemical potentials coming from the differences $\omega_1 - \omega_2$, $\omega_1 - \Omega_1$, $\Omega_1 - \Omega_2$ and $\Omega_2 - \Omega_3$. These four effective chemical potentials should then correspond to turning on four magnetic fields in the $SU(1,2|3)$ spin chain, and furthermore correspond to having four rotation angles on the dual pp-wave background.

A particularly important aspect of the decoupling limit (2.13) is that it could allow to directly investigate the validity of the AdS/CFT correspondence. This is realized by the fact that on the gauge theory side we can take a strong $\tilde{\lambda}$ limit even though the 't Hooft coupling λ goes to zero in the limit (2.13). The strong $\tilde{\lambda}$ regime should then be related via the AdS/CFT correspondence to the string theory dual of the gauge theory under investigation. This means that we can study weakly coupled gauge theory and string theory in the same regime and thus we can hope to compare the computations on both sides directly.

As explained in the main text, the decoupling limit (2.13) is defined also for finite values of N, N being the number of colors. Thus, using the decoupling limit (2.13) one can obtain a very convenient environment where to compute the non-planar corrections to the gauge theory partition function. We expect that this will allow one to gain more information about important aspects of the Hagedorn/deconfinement phase transition. For example it should then be possible to study interesting questions such as what the order of the phase transition is or what the nature of the phase above the Hagedorn transition is, and one could furthermore hope to understand the behavior of the theory for very high temperatures both at weak and strong coupling $\tilde{\lambda}$. Employing the fact that our decoupling limits work for finite N we can also hope to understand effects for black holes in $AdS_5 \times S^5$. This could potentially lead to a better understanding of such important issues as the unitarity of black hole physics and the microstates of black holes.

Finally, it would be interesting to generalize our results to other gauge theories. In particular, it would be interesting to study decoupling limits of thermal $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3/\mathbb{Z}_k$ and of the dimensionally reduced 2+1 dimensional SYM theory on $\mathbb{R} \times S^2$ [33–37]. This would be interesting in view of the effects of the non-trivial vacua, and here a decoupling limit of the kind presented in this paper could be essential to study the theories beyond the zero-coupling regime.

Acknowledgments

A. Oscillator representation of $u(2,2|4)$

The $\mathcal{N} = 4$ SYM theory has $P\,\mathrm{SU}(2,2|4)$ as global symmetry. Since we use the algebraic characterizations of the decoupled theories extensively in the main text, we review in this appendix the oscillator representation of $u(2,2|4)$ [25, 12], which is a highly useful way of representing both the algebra and the set of letters of $\mathcal{N} = 4$ SYM. In appendix B we use this to understand the algebra and representation for each decoupled theory.

The generators of $u(2,2|4)$. In the oscillator representation of $u(2,2|4)$ we consider two bosonic oscillators $\mathbf{a}^\alpha$, $\mathbf{b}^{\dot\alpha}$, $\alpha, \dot\alpha = 1, 2$, and one fermionic oscillator $\mathbf{c}^a$, $a = 1, 2, 3, 4$, with the commutation relations [25, 12]

$$\left[\mathbf{a}^\alpha, \mathbf{a}^\dagger_\beta\right] = \delta^\alpha_\beta \,, \quad \left[\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}\right] = \delta^{\dot\alpha}_{\dot\beta} \,, \quad \left\{\mathbf{c}^a, \mathbf{c}^\dagger_b\right\} = \delta^a_b \,. \tag{A.1}$$

Define furthermore the number operators

$$a^\alpha = \mathbf{a}^\dagger_\alpha \mathbf{a}^\alpha \,, \quad b^{\dot\alpha} = \mathbf{b}^\dagger_{\dot\alpha} \mathbf{b}^{\dot\alpha} \,, \quad c^a = \mathbf{c}^\dagger_a \mathbf{c}^a \tag{A.2}$$

where we do not sum over the indices, and we have $\alpha, \dot\alpha = 1, 2$ and $a = 1, 2, 3, 4$.

In terms of the oscillators the $so(4) \simeq su(2) \times su(2)$ subalgebra of $u(2,2|4)$ has the 6 generators given by

$$\mathbf{L}^\alpha_{\ \beta} = \mathbf{a}^\dagger_\beta \mathbf{a}^\alpha - \frac{a^1 + a^2}{2}\delta^\alpha_\beta \,, \quad \dot{\mathbf{L}}^{\dot\alpha}_{\ \dot\beta} = \mathbf{b}^\dagger_{\dot\beta} \mathbf{b}^{\dot\alpha} - \frac{b^1 + b^2}{2}\delta^{\dot\alpha}_{\dot\beta} \,. \tag{A.3}$$

The 15 generators of the $su(4)$ subalgebra are

$$\mathbf{R}^a_{\ b} = \mathbf{c}^\dagger_b \mathbf{c}^a - \frac{1}{4}\delta^a_b \sum_{d=1}^4 c^d \,. \tag{A.4}$$

We have three $u(1)$ charges being the bare dilatation operator[8] D_0, the central charge C and the hypercharge B, given as

$$\begin{aligned} D_0 &= 1 + \frac{1}{2}(a^1 + a^2 + b^1 + b^2) \\ C &= 1 + \frac{1}{2}(-a^1 - a^2 + b^1 + b^2 - c^1 - c^2 - c^3 - c^4) \\ B &= \frac{1}{2}(a^1 + a^2 - b^1 - b^2). \end{aligned} \tag{A.5}$$

In addition to this, we have four translation generators $\mathbf{P}_{\alpha\dot\beta}$ and four boost generators $\mathbf{K}^{\alpha\dot\beta}$ given by

$$\mathbf{P}_{\alpha\dot\beta} = \mathbf{a}^\dagger_\alpha \mathbf{b}^\dagger_{\dot\beta} \,, \quad \mathbf{K}^{\alpha\dot\beta} = \mathbf{a}^\alpha \mathbf{b}^{\dot\beta} \tag{A.6}$$

and the 32 fermionic generators

$$\mathbf{Q}^a_{\ \alpha} = \mathbf{a}^\dagger_\alpha \mathbf{c}^a \,, \quad \dot{\mathbf{Q}}_{\dot\alpha a} = \mathbf{b}^\dagger_{\dot\alpha} \mathbf{c}^\dagger_a \,, \quad \mathbf{S}^\alpha_{\ a} = \mathbf{c}^\dagger_a \mathbf{a}^\alpha \,, \quad \dot{\mathbf{S}}^{\dot\alpha a} = \mathbf{b}^{\dot\alpha} \mathbf{c}^a. \tag{A.7}$$

---

[8]Note that here we are concerned with the $psu(2,2|4)$ algebra of $\mathcal{N} = 4$ SYM for zero gauge coupling, *i.e.* $\lambda = 0$.

The set of 32 bosonic generators ($\mathbf{R}^a{}_b$, $\mathbf{L}^\alpha{}_\beta$, $\dot{\mathbf{L}}^{\dot\alpha}{}_{\dot\beta}$, D_0, C, B, $\mathbf{P}_{\alpha\dot\beta}$ and $\mathbf{K}^{\alpha\dot\beta}$) and the 32 fermionic generators ($\mathbf{Q}^a{}_\alpha$, $\dot{\mathbf{Q}}_{\dot\alpha a}$, $\mathbf{S}^\alpha{}_a$ and $\dot{\mathbf{S}}^{\dot\alpha a}$) together comprise the algebra of $u(2,2|4)$. The commutation relations can be worked out explicitly using the commutation relations (A.1) for the oscillators.

One can consistently drop the hypercharge B from the $u(2,2|4)$ algebra, revealing $su(2,2|4)$. If one sticks to representations with $C = 0$, one can furthermore take out C of the algebra, which means going from $su(2,2|4)$ to the $psu(2,2|4)$ algebra that is the algebra for the global symmetries of $\mathcal{N} = 4$ SYM.

It is useful to connect here the Cartan subalgebra of $u(2,2|4)$ in terms of the oscillator representation to the notation that we employ in the main text. In addition to the three $u(1)$ charges D_0, C and B defined in (A.5), we have the following Cartan generators of the $so(4) \simeq su(2) \times su(2)$ algebra

$$S_L = \frac{1}{2}(a^1 - a^2), \quad S_R = \frac{1}{2}(b^1 - b^2) \tag{A.8}$$

with the relation that $S_L = (S_1 - S_2)/2$ and $S_R = (S_1 + S_2)/2$, S_1 and S_2 being the $so(4)$ Cartan generators. The $so(4)$ Cartan generators are thus

$$S_1 = \frac{1}{2}(a^1 - a^2 + b^1 - b^2), \quad S_2 = \frac{1}{2}(-a^1 + a^2 + b^1 - b^2). \tag{A.9}$$

The Cartan generators we use for $su(4)$ are

$$J_1 = \frac{1}{2}(-c^1 - c^2 + c^3 + c^4), \quad J_2 = \frac{1}{2}(-c^1 + c^2 - c^3 + c^4), \quad J_3 = \frac{1}{2}(c^1 - c^2 - c^3 + c^4). \tag{A.10}$$

Since $u(2,2|4)$ has fermionic generators it is not unique how to split up the generators into raising and lowering operators. The choice we use in almost all cases is the one dubbed the "Beauty" in [16] and corresponds to choosing $\mathbf{S}^\alpha{}_a$ and $\dot{\mathbf{S}}^{\dot\alpha a}$ as the fermionic raising operators. The Dynkin diagram of the "Beauty" is

$$\bigcirc\!\!-\!\!\otimes\!\!-\!\!\bigcirc\!\!-\!\!\bigcirc\!\!-\!\!\bigcirc\!\!-\!\!\otimes\!\!-\!\!\bigcirc \tag{A.11}$$

Here the $\bigcirc$ refers to a bosonic root, while $\otimes$ refers to a fermionic root. We note that up to an overall sign the Cartan matrix M is uniquely determined by the Dynkin diagram (see for example [38] for the rules of constructing the Cartan matrix). The lowering operators of $u(2,2|4)$ corresponding to minus the simple roots associated with the Dynkin diagram (A.11) are

$$\mathbf{a}_1^\dagger \mathbf{a}^2, \quad \mathbf{a}_2^\dagger \mathbf{c}^1, \quad \mathbf{c}_1^\dagger \mathbf{c}^2, \quad \mathbf{c}_2^\dagger \mathbf{c}^3, \quad \mathbf{c}_3^\dagger \mathbf{c}^4, \quad \mathbf{b}_2^\dagger \mathbf{c}_4^\dagger, \quad \mathbf{b}_1^\dagger \mathbf{b}^2. \tag{A.12}$$

We see that the three bosonic roots in the middle of (A.11) correspond to the $su(4)$ R-symmetry algebra. We choose the diagonal of the Cartan matrix to be positive for these

three roots. With this, the Cartan matrix is

$$
M = \begin{pmatrix}
-2 & +1 & & & & & \\
+1 & & -1 & & & & \\
& -1 & +2 & -1 & & & \\
& & -1 & +2 & -1 & & \\
& & & -1 & +2 & -1 & \\
& & & & -1 & & +1 \\
& & & & & +1 & -2
\end{pmatrix} . \tag{A.13}
$$

The Dynkin labels corresponding to (A.11) are

$$[s_1, r_1, q_1, p, q_2, r_2, s_2] \tag{A.14}$$

with $s_1 = a^2 - a^1$ and $s_2 = b^2 - b^1$ corresponding to the $su(2) \times su(2)$ subgroups, $q_1 = c^2 - c^1$, $p = c^3 - c^2$ and $q_2 = c^4 - c^3$ corresponding to the $su(4)$ subgroup, and finally for the two fermionic roots we have $r_1 = a^2 + c^1$ and $r_2 = 1 + b^2 - c^4$.

Another choice for the raising and lowering operators is the one dubbed the "Beast" in [16] and corresponds to choosing $\mathbf{S}^{\alpha}{}_a$ and $\dot{\mathbf{Q}}_{\dot{\alpha}a}$ as the fermionic raising operators. This choice is useful for the SU(1,2) decoupled theory. The Dynkin diagram is

$$\bigcirc\!\!-\!\!\bigcirc\!\!-\!\!\bigcirc\!\!-\!\!\otimes\!\!-\!\!\bigcirc\!\!-\!\!\bigcirc\!\!-\!\!\bigcirc \tag{A.15}$$

The lowering operators corresponding to minus the simple roots associated with the Dynkin diagram are

$$\mathbf{a}_1^{\dagger}\mathbf{a}^2 , \quad \mathbf{a}_2^{\dagger}\mathbf{b}_2^{\dagger} , \quad \mathbf{b}_1^{\dagger}\mathbf{b}^2 , \quad \mathbf{b}^1\mathbf{c}^1 , \quad \mathbf{c}_1^{\dagger}\mathbf{c}^2 , \quad \mathbf{c}_2^{\dagger}\mathbf{c}^3 , \quad \mathbf{c}_3^{\dagger}\mathbf{c}^4 . \tag{A.16}$$

The first three roots of (A.15) correspond to the $su(2,2)$ subalgebra. We choose the Cartan matrix to be positive in the diagonal for these nodes. The Dynkin labels for the three nodes of $su(2,2)$ are $[s_1, r, s_2]$ with $r = -1 - a^2 - b^2$.

The letters of $\mathcal{N} = 4$ SYM. In section 2 we described the set of letters $\mathcal{A}$ of $\mathcal{N} = 4$ SYM. The letters are listed in tables 1–5 along with the four components of the covariant derivative in table 6 using which one obtain the descendants. In terms of the oscillators $\mathbf{a}^{\alpha}$, $\mathbf{b}^{\dot{\alpha}}$, $\alpha, \dot{\alpha} = 1, 2$, and $\mathbf{c}^a$, $a = 1, 2, 3, 4$, the set of letters $\mathcal{A}$ of $\mathcal{N} = 4$ SYM is given by

$$
\begin{aligned}
\phi \; &: \; (\mathbf{c}^{\dagger})^2|0\rangle \; \text{repr.} \; [0,1,0]_{(0,0)} \\
\chi \; &: \; \mathbf{a}^{\dagger}\mathbf{c}^{\dagger}|0\rangle \; \text{repr.} \; [0,0,1]_{(\frac{1}{2},0)} , \quad \bar{\chi} \; : \; \mathbf{b}^{\dagger}(\mathbf{c}^{\dagger})^3|0\rangle \; \text{repr.} \; [1,0,0]_{(0,\frac{1}{2})} \\
F \; &: \; (\mathbf{a}^{\dagger})^2|0\rangle \; \text{repr.} \; [0,0,0]_{(1,0)} , \quad \bar{F} \; : \; (\mathbf{b}^{\dagger})^2(\mathbf{c}^{\dagger})^4|0\rangle \; \text{repr.} \; [0,0,0]_{(0,1)} \\
d \; &: \; \mathbf{a}^{\dagger}\mathbf{b}^{\dagger} \; \text{repr.} \; [0,0,0]_{(\frac{1}{2},\frac{1}{2})}
\end{aligned} \tag{A.17}
$$

It is an easy exercise to find the explicit oscillator representation for each letter of $\mathcal{N} = 4$ SYM by combining (A.17) with the Cartan generators (A.9) and (A.10) for the $so(4)$ and $su(4)$ algebras and with the tables 1–6 of the letters.

All the letters of $\mathcal{N} = 4$ SYM have $C = 0$ thus the set of letters $\mathcal{A}$ corresponds to a representation of $psu(2,2|4)$. Considering the "Beauty" (A.11) one can see from the

hermitian conjugate of (A.12) that the letter $Z = \mathbf{c}_3^\dagger \mathbf{c}_4^\dagger |0\rangle$ is the highest weight of the representation. Therefore, using the Dynkin labels (A.14), we see that the set of letters $\mathcal{A}$ corresponds to the $[0, 0, 0, 1, 0, 0, 0]$ representation of $psu(2, 2|4)$. This representation is known as the Singleton representation.

B. Algebras and representations for decoupled theories

We describe in this appendix briefly the algebras and representations for each of the twelve non-trivial decoupled theories. This is done in terms of the oscillator representation of $u(2, 2|4)$ reviewed in appendix A where also the notation used below is defined. Each decoupled theory corresponds to a sector of $\mathcal{N} = 4$ SYM, i.e. a subset of the full set of letters with a sub-algebra of the full algebra $psu(2, 2|4)$. The algebras and representations can be derived from the "Beauty" Dynkin diagram (A.11), except for the SU(1, 2) decoupled theory which is derived from the "Beast" Dynkin diagram (A.15).

The SU(2) sector is given by

$$c^1 = 0, \quad c^4 = 1, \quad a^1 = a^2 = b^1 = b^2 = 0. \tag{B.1}$$

From the Dynkin diagram (A.11) of the full $psu(2, 2|4)$ algebra we keep only the $\mathbf{c}_2^\dagger \mathbf{c}^3$ node which has Dynkin label $[p] = [c^3 - c^2]$. The highest weight within this subsector is Z which gives $[p] = [1]$. This is twice the spin which fits with this being the spin 1/2 representation.

The bosonic SU(1, 1) sector is defined by

$$c^1 = c^2 = 0, \quad c^3 = c^4 = 1, \quad a^2 = b^2 = 0. \tag{B.2}$$

The Dynkin diagram has one bosonic node with

$$\mathbf{a}_1^\dagger \mathbf{b}_1^\dagger = (\mathbf{a}_1^\dagger \mathbf{a}^2)(\mathbf{a}_2^\dagger \mathbf{b}_2^\dagger)(\mathbf{b}_1^\dagger \mathbf{b}^2) \tag{B.3}$$

as the lowering operator. The Dynkin label is $[r'] = [-1 - a^1 - b^1]$ which for Z gives $[-1]$. This is again twice the spin of the representation which fits with this being the spin $-1/2$ representation.

The fermionic SU(1, 1) sector is given by

$$c^1 = c^2 = c^3 = 0, \quad c^4 = 1, \quad a^2 = b^2 = 0. \tag{B.4}$$

The Dynkin diagram is the same as for the bosonic SU(1, 1) case but in this sector the fermion χ_1 is the highest weight and the Dynkin label becomes $[r'] = [-2]$ which fits well with this being the spin -1 representation.

The SU(1|1) sector is given by

$$c^1 = c^2 = 0, \quad c^4 = 1, \quad a^2 = b^1 = b^2 = 0. \tag{B.5}$$

The Dynkin diagram has one fermionic node with

$$\mathbf{a}_1^\dagger \mathbf{c}^3 = (\mathbf{a}_1^\dagger \mathbf{a}^2)(\mathbf{a}_2^\dagger \mathbf{c}^1)(\mathbf{c}_1^\dagger \mathbf{c}^2)(\mathbf{c}_2^\dagger \mathbf{c}^3) \tag{B.6}$$

as the simple lowering operator. The Dynkin label is $[r_1'] = [a^1 + c^3]$ and for the highest weight Z we get $[r_1'] = [1]$.

The SU(1|2) sector is defined by

$$c^1 = 0, \quad c^4 = 1, \quad a^1 = a^2 = b^2 = 0. \tag{B.7}$$

This is our first example of a symmetry algebra with rank higher than one. The Dynkin diagram has in this case two nodes, one bosonic and one fermionic

$$\bigcirc\!\!-\!\!\otimes \tag{B.8}$$

The lowering operators corresponding to these nodes are

$$\mathbf{c}_2^\dagger \mathbf{c}^3, \quad \mathbf{b}_1^\dagger \mathbf{c}_3^\dagger = (\mathbf{c}_3^\dagger \mathbf{c}^4)(\mathbf{b}_2^\dagger \mathbf{c}_4^\dagger)(\mathbf{b}_1^\dagger \mathbf{b}^2). \tag{B.9}$$

The Dynkin labels are $[p, r_2'] = [c^3 - c^2, 1 + b^1 - c^3]$ which for the highest weight Z gives $[1, 0]$.

The SU(2|3) sector is defined by

$$c^4 = 1, \quad b^1 = b^2 = 0. \tag{B.10}$$

For this sector we keep the first four nodes of the Beauty diagram

$$\bigcirc\!\!-\!\!\otimes\!\!-\!\!\bigcirc\!\!-\!\!\bigcirc \tag{B.11}$$

with lowering operators

$$\mathbf{a}_1^\dagger \mathbf{a}^2, \quad \mathbf{a}_2^\dagger \mathbf{c}^1, \quad \mathbf{c}_1^\dagger \mathbf{c}^2, \quad \mathbf{c}_2^\dagger \mathbf{c}^3. \tag{B.12}$$

The Dynkin labels for this sector are $[s_1, r_1, q_1, p]$ which for the highest weight Z gives $[0, 0, 0, 1]$.

The SU(1, 1|1) sector is defined by

$$c^1 = 0, \quad c^3 = c^4 = 1, \quad a^2 = b^2 = 0. \tag{B.13}$$

We can obtain all these states from the highest weight Z by combining the first three lowering operators of the Beauty into one fermionic operator

$$\mathbf{a}_1^\dagger \mathbf{c}^2 = (\mathbf{a}_1^\dagger \mathbf{a}^2)(\mathbf{a}_2^\dagger \mathbf{c}^1)(\mathbf{c}_1^\dagger \mathbf{c}^2) \tag{B.14}$$

and by combining the last four lowering operators into another fermionic operator

$$\mathbf{b}_1^\dagger \mathbf{c}_2^\dagger = (\mathbf{c}_2^\dagger \mathbf{c}^3)(\mathbf{c}_3^\dagger \mathbf{c}^4)(\mathbf{b}_2^\dagger \mathbf{c}_4^\dagger)(\mathbf{b}_1^\dagger \mathbf{b}^2). \tag{B.15}$$

Since this sector has two fermionic roots the Dynkin diagram is

$$\otimes\!\!-\!\!\otimes \tag{B.16}$$

and the Dynkin labels are $[r_1', r_2'] = [a^1 + c^2, 1 + b^1 - c^2]$ which for the highest weight Z gives $[0, 1]$.

The SU(1, 1|2) sector is defined by

$$c^1 = 0, \quad c^4 = 1, \quad a^2 = b^2 = 0. \tag{B.17}$$

We use a similar combination of roots as in the previous sector, except now we keep the middle operator as it is in the Beauty. The three lowering operators that we have at our disposal are then

$$\mathbf{a}_1^\dagger \mathbf{c}^2 = (\mathbf{a}_1^\dagger \mathbf{a}^2)(\mathbf{a}_2^\dagger \mathbf{c}^1)(\mathbf{c}_1^\dagger \mathbf{c}^2), \qquad \mathbf{c}_2^\dagger \mathbf{c}^3, \qquad \mathbf{b}_1^\dagger \mathbf{c}_3^\dagger = (\mathbf{c}_3^\dagger \mathbf{c}^4)(\mathbf{b}_2^\dagger \mathbf{c}_4^\dagger)(\mathbf{b}_1^\dagger \mathbf{b}^2). \tag{B.18}$$

The Dynkin diagram is

$$\otimes\!\!-\!\!\bigcirc\!\!-\!\!\otimes \tag{B.19}$$

and the Dynkin labels are $[r_1', p, r_2'] = [a^1 + c^2, c^3 - c^2, 1 + b^1 - c^3] = [0, 1, 0]$

The SU(1, 2) sector is defined by

$$c^1 = c^2 = c^3 = c^4 = 1, \quad b^2 = 0. \tag{B.20}$$

As already mentioned, we obtain this sector from the "Beast" Dynkin diagram (A.15). Specifically, we consider the first three nodes of (A.15) corresponding to the $su(2, 2)$ algebra. We keep the first node as it is but combine the latter two. The lowering operators that we get in this way are

$$\mathbf{a}_1^\dagger \mathbf{a}^2, \qquad \mathbf{a}_2^\dagger \mathbf{b}_1^\dagger = (\mathbf{a}_2^\dagger \mathbf{b}_2^\dagger)(\mathbf{b}_1^\dagger \mathbf{b}^2). \tag{B.21}$$

The Dynkin diagram is

$$\bigcirc\!\!-\!\!\bigcirc \tag{B.22}$$

and the Dynkin labels are $[s_1, r'] = [a^2 - a^1, -1 - a^2 - b^1] = [0, -3]$.

The SU(1, 2|1) sector is defined by

$$c^2 = c^3 = c^4 = 1, \quad b^2 = 0. \tag{B.23}$$

We need three lowering operators

$$\mathbf{a}_1^\dagger \mathbf{a}^2, \quad \mathbf{a}_2^\dagger \mathbf{c}^1, \quad \mathbf{b}_1^\dagger \mathbf{c}_1^\dagger = (\mathbf{c}_1^\dagger \mathbf{c}^2)(\mathbf{c}_2^\dagger \mathbf{c}^3)(\mathbf{c}_3^\dagger \mathbf{c}^4)(\mathbf{b}_2^\dagger \mathbf{c}_4^\dagger)(\mathbf{b}_1^\dagger \mathbf{b}^2). \tag{B.24}$$

The Dynkin diagram is

$$\bigcirc\!\!-\!\!\otimes\!\!-\!\!\otimes \tag{B.25}$$

and the Dynkin labels are $[s_1, r_1, r_2'] = [a^2 - a^1, a^2 + c^1, 1 + b^1 - c^1] = [0, 0, 2]$.

The SU(1, 2|2) sector is given by

$$c^3 = c^4 = 1, \quad b^2 = 0. \tag{B.26}$$

We need four lowering operators

$$\mathbf{a}_1^\dagger \mathbf{a}^2, \qquad \mathbf{a}_2^\dagger \mathbf{c}^1, \qquad \mathbf{c}_1^\dagger \mathbf{c}^2, \qquad \mathbf{b}_1^\dagger \mathbf{c}_2^\dagger = (\mathbf{c}_2^\dagger \mathbf{c}^3)(\mathbf{c}_3^\dagger \mathbf{c}^4)(\mathbf{b}_2^\dagger \mathbf{c}_4^\dagger)(\mathbf{b}_1^\dagger \mathbf{b}^2). \qquad (B.27)$$

Starting from Z we can get all the other allowed letters in this sector by applying these four lowering operators in a particular order. First let us see how to go from Z to any of the other types of fields:

$$Z \;\to\; \mathbf{b}_1^\dagger \mathbf{c}_2^\dagger \, Z = \bar\chi_7 \;\to\; \mathbf{c}_1^\dagger \mathbf{c}^2 \, \bar\chi_7 = \bar\chi_5 \;\to\; \mathbf{b}_1^\dagger \mathbf{c}_2^\dagger \, \bar\chi_5 = \bar F_+ \,. \qquad (B.28)$$

We can also obtain $d_1 Z$ or $d_2 Z$ from plain Z:

$$Z \;\to\; \mathbf{b}_1^\dagger \mathbf{c}_2^\dagger \, Z = \bar\chi_7 \;\to\; \mathbf{c}_1^\dagger \mathbf{c}^2 \, \bar\chi_7 = \bar\chi_5 \;\to\; \mathbf{a}_2^\dagger \mathbf{c}^1 \, \bar\chi_5 = d_2 Z \;\to\; \mathbf{a}_1^\dagger \mathbf{a}^2 \, d_2 Z = d_1 Z. \quad (B.29)$$

Using the second chain repeatedly we can clearly get $d_1^k d_2^\ell Z$ and using the first chain we can map $d_1^k d_2^\ell Z$ to any of the other letters with the same set of derivatives. The Dynkin diagram is

$$\bigcirc\!\!-\!\!\otimes\!\!-\!\!\bigcirc\!\!-\!\!\otimes \qquad (B.30)$$

and the Dynkin labels are $[s_1, r_1, q_1, r_2']$ with $r_2' = 1 + b^1 - c^2$. The highest weight is Z and we get $[0,0,0,1]$.

The $SU(1,2|3)$ sector is defined by

$$c^4 = 1, \quad b^2 = 0. \qquad (B.31)$$

We need five lowering operators and we get them from the Beauty by combining the last three roots into one fermionic root. The corresponding operator will be $\mathbf{b}_1^\dagger \mathbf{c}_3^\dagger = (\mathbf{c}_3^\dagger \mathbf{c}^4)(\mathbf{b}_2^\dagger \mathbf{c}_4^\dagger)(\mathbf{b}_1^\dagger \mathbf{b}^2)$ with Dynkin label $r_2' = 1 + b^1 - c^3$. The highest weight is Z and the Dynkin labels are $[s_1, r_1, q_1, p, r_2'] = [0,0,0,1,0]$. The Dynkin diagram is

$$\bigcirc\!\!-\!\!\otimes\!\!-\!\!\bigcirc\!\!-\!\!\bigcirc\!\!-\!\!\otimes \qquad (B.32)$$

The Cartan matrices for all our decoupled theories can be obtained from the Cartan matrix of the "Beauty" in eq. (A.13) by deleting appropriate columns and rows in accordance with the Dynkin diagram of each sector. A fermionic node always gives rise to a zero on the diagonal while bosonic roots give either plus or minus two. This is with the notable exception of the $SU(1,2)$ theory for which we have the usual Cartan matrix of the $sl(3)$ algebra.

C. The letter partition function

The result (4.17) for the letter partition function for $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the presence of non-zero chemical potentials for the R-charges of the SU(4) R-symmetry and for the Cartan generators of the SO(4) symmetry group of S^3 can be also obtained using the

oscillator picture. In this formalism, the general expression for the letter partition function is given by

$$z(x, \omega_i, y_i) = \text{Tr}_{\mathcal{A}} \left(x^{D_0} y_1^{J_1} y_2^{J_2} y_3^{J_3} \rho_1^{S_1} \rho_2^{S_2} \right) \tag{C.1}$$

$$= \sum_{a^1, a^2, b^1, b^2 = 0}^{\infty} \sum_{c^1, c^2, c^3, c^4 = 0}^{1} \delta(C) x^{D_0} y_1^{J_1} y_2^{J_2} y_3^{J_3} \rho_1^{S_1} \rho_2^{S_2}$$

$$= \sum_{a^1, a^2 = 0}^{\infty} \sum_{n^1 = -a^1}^{\infty} \sum_{n^2 = -a^2}^{\infty} \left[Y_S \delta \left(\frac{n^1 + n^2}{2} \right) + \delta \left(1 + \frac{n^1 + n^2}{2} \right) + \delta \left(\frac{n^1 + n^2}{2} - 1 \right) \right.$$

$$\left. + Y_1 \delta \left(\frac{1 + n^1 + n^2}{2} \right) + Y_2 \delta \left(\frac{n^1 + n^2 - 1}{2} \right) \right] x^{1 + \frac{n^1 + n^2}{2} + a^1 + a^2} \rho_1^{\frac{n^1 - n^2}{2} + a^1 - a^2} \rho_2^{\frac{n^1 - n^2}{2}}$$

where $n^k = b^k - a^k$, $k = 1, 2$, $y_i = \exp(\beta \Omega_i)$, $i = 1, 2, 3$, $\rho_j = \exp(\beta \omega_j)$, $j = 1, 2$,

$$Y_S = \sum_{i=1}^{3} (y_i + y_i^{-1})$$

$$Y_1 = (y_1 y_2 y_3)^{1/2} + y_1^{1/2} (y_2 y_3)^{-1/2} + (y_1 y_3)^{-1/2} y_2^{1/2} + (y_1 y_2)^{-1/2} y_3^{1/2} \tag{C.2}$$

$$Y_2 = (y_1 y_2 y_3)^{-1/2} + y_1^{-1/2} (y_2 y_3)^{1/2} + (y_1 y_3)^{1/2} y_2^{-1/2} + (y_1 y_2)^{1/2} y_3^{-1/2} \tag{C.3}$$

In the second line we performed the sums over the fermionic operators c^a, $a = 1, 2, 3, 4$. From equation (C.1) it is easy to see that, by performing all the sums, we can derive the contribution of scalars, vectors and fermions in the various representation separately. In more detail, the term proportional to $\delta \left(\frac{n^1 + n^2}{2} \right)$ gives the scalar partition function (4.7). This can be also seen from the fact that the scalars are given acting with $(\mathbf{c}^\dagger)^2$ on the vacuum and the delta function for the central charge is given by $\delta \left(\frac{n^1 + n^2}{2} \right)$ precisely when $\sum_{a=1}^{4} c^a = 2$.

The term proportional to $\delta \left(1 + \frac{n^1 + n^2}{2} \right)$ gives the partition function for vectors in the representation $[0, 0, 0]_{(1,0)}$. In fact this is the contribution when $\sum_{a=1}^{4} c^a = 0$. For $\sum_{a=1}^{4} c^a = 4$ we get instead the term proportional to $\delta \left(\frac{n^1 + n^2}{2} - 1 \right)$ that corresponds to the partition function for vectors in the representation $[0, 0, 0]_{(0,1)}$. Adding together the two contributions we get the result (4.10) for the vectors partition function.

By similar arguments, the term proportional to $\delta \left(\frac{1 + n^1 + n^2}{2} \right)$ gives the partition function (4.13) for fermions in the representation $[0, 0, 1]_{(1/2, 0)}$ and, finally, the term proportional to $\delta \left(\frac{n^1 + n^2 - 1}{2} \right)$ gives the partition function (4.16) for fermions in the representation $[1, 0, 0]_{(0, 1/2)}$.

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